ISRAEL JOURNAL OF MATHEMATICS **158** (2007), 65–104
DOI: 10.1007/s11856-007-0004-z

THREE TYPES OF INCLUSIONS OF INNATELY TRANSITIVE PERMUTATION GROUPS INTO WREATH PRODUCTS IN PRODUCT ACTION

BY

Cheryl E. Praeger

School of Mathematics and Statistics, The University of Western Australia
35 Stirling Highway, Crawley, Western Australia 6009
e-mail: praeger@maths.uwa.edu.au
URL: www.maths.uwa.edu.au/~praeger

AND

Csaba Schneider

Informatics Research Laboratory, Computer and Automation Research Institute
1518 Budapest, Pf. 63, Hungary
e-mail: csaba.schneider@sztaki.hu
URL: www.sztaki.hu/~schneider

ABSTRACT

A permutation group is innately transitive if it has a transitive minimal normal subgroup, and this subgroup is called a plinth. In this paper we study three special types of inclusions of innately transitive permutation groups in wreath products in product action. This is achieved by studying the natural Cartesian decomposition of the underlying set that corresponds to the product action of the wreath product. Previously we identified six classes of Cartesian decompositions that can be acted upon transitively by an innately transitive group with a non-abelian plinth. The inclusions studied in this paper correspond to three of the six classes. We find that in each case the isomorphism type of the acting group is restricted, and some interesting combinatorial structures are left invariant. We also give a fairly general construction of inclusions for each type.

Received August 3, 2004

1. Introduction

The various versions of the O'Nan–Scott Theorem identify several classes of finite primitive [LPS88], quasiprimitive [Pra93], and innately transitive [BamP04] permutation groups and give a description of the groups in each of the classes. (A permutation group is called **quasiprimitive** if each of its minimal normal subgroups is transitive, and it is called **innately transitive** if it has at least one minimal normal subgroup that is transitive.) In several combinatorial and group theoretic applications, the sketchy descriptions in the O'Nan–Scott theorems were not sufficient, and further information about some of the families was needed before the theorem could be used effectively. In particular, in all variants of the theorem at least one of the classes of groups is formed by certain subgroups of wreath products. A sufficiently detailed description of inclusions of finite primitive groups into wreath products was achieved in [Kov89a, Pra90]. Our program aims for an equally satisfactory description of inclusions of finite innately transitive, and hence also of quasiprimitive, groups into wreath products in their product action.

Deciding whether a subgroup of a finite wreath product in product action is innately transitive can be checked by exhibiting a transitive minimal normal subgroup. Our problem is the reverse: given an innately transitive permutation group, determine whether it can be a subgroup of a wreath product in product action, and if so, describe all such wreath products. Some inclusions of innately transitive groups into wreath products are natural and can easily be identified. However, some others are not quite so apparent. In Sections 1 and 2 of [BPS06], we give several different kinds of examples. We found the existence of some of these examples rather surprising.

Suppose that G is a finite innately transitive subgroup of $\mathsf{Sym}\,\Omega$. Our aim is to find all inclusions of G into subgroups $W \cong \mathsf{Sym}\,\Gamma\,\mathsf{wr}\,\mathsf{S}_\ell$ of $\mathsf{Sym}\,\Omega$ such that G projects onto a transitive subgroup of S_ℓ. Here the group W is considered as a permutation group acting on Γ^ℓ in product action. (The case in which the projection of G is intransitive is dealt with in [BPSxx].) Such problems arise in algebraic combinatorics where we are often given a combinatorial structure with a subgroup of its automorphism group; our task is to determine a larger subgroup of the automorphism group, or, where possible, the full automorphism group itself. The case where the given group preserves additional structure on points, such as a Cartesian decomposition (as studied in this paper), is often difficult to identify as its existence may not be apparent from the given combinatorial information.

If G is contained in a wreath product W as above then the underlying set Ω can be identified with the Cartesian product Γ^ℓ, such that the groups G and W preserve the natural Cartesian decomposition of Γ^ℓ (see Section 2). Moreover, the permutation representation of G induced by the natural projection $W \to \mathsf{S}_\ell$ is equivalent to the G-action on this Cartesian decomposition. In [6-Class Theorem, BPS06] we identified six pairwise disjoint classes of Cartesian decompositions acted upon transitively by an innately transitive group with a non-abelian transitive minimal normal subgroup. The names of these classes are $\mathsf{CD}_{\rm S}(G)$, $\mathsf{CD}_1(G)$, $\mathsf{CD}_{1\rm S}(G)$, $\mathsf{CD}_{2\sim}(G)$, $\mathsf{CD}_{2\not\sim}(G)$ and $\mathsf{CD}_3(G)$ (see Section 2). A G-invariant Cartesian decomposition of Ω in a particular class leads to a special type of embedding of G into a wreath product in product action. Cartesian decompositions in $\mathsf{CD}_{\rm S}(G)$ and $\mathsf{CD}_1(G)$ were described in [BPS06], while $\mathsf{CD}_3(G)$ was studied in [PS03]. The aim of this paper is to investigate the remaining three classes, namely $\mathsf{CD}_{1\rm S}(G)$, $\mathsf{CD}_{2\sim}(G)$ and $\mathsf{CD}_{2\not\sim}(G)$.

We believe that the classes $\mathsf{CD}_{1\rm S}(G)$, $\mathsf{CD}_{2\sim}(G)$, $\mathsf{CD}_{2\not\sim}(G)$ and $\mathsf{CD}_3(G)$ are the most challenging ones of the 6-Class Theorem. The classes $\mathsf{CD}_{\rm S}(G)$, $\mathsf{CD}_1(G)$ can be viewed as natural. Indeed, they arise from "normal" Cartesian decompositions preserved by G (as defined in [BPSyy]). On the other hand, the remaining classes correspond to exceptional embeddings of innately transitive groups into wreath products. The aim of the research presented here is to understand the exceptional embeddings that correspond to a Cartesian decomposition in $\mathsf{CD}_{1\rm S}(G)$, $\mathsf{CD}_{2\sim}(G)$ or $\mathsf{CD}_{2\not\sim}(G)$. We describe these in as much detail as feasible in our framework. Thus this paper contains three main results that are proved in Sections 5, 6, and 7.

As we do not want to litter the introduction with complicated notation, instead of explicitly stating our main results here, we present the following schema on which Theorems 5.1, 6.1 and 7.1 are built. Let G be an innately transitive group with transitive minimal normal subgroup M, where M is isomorphic to the direct power of a non-abelian, finite simple group T. Suppose that $\mathcal{E} \in \mathsf{CD}_{2\sim}(G) \cup \mathsf{CD}_{2\not\sim}(G) \cup \mathsf{CD}_{1\rm S}(G)$. Then in each case we prove three properties of the permutation group G and the Cartesian decomposition $\mathcal{E}$.

[1] (Quotient Action Property) We study G via its action on a G-invariant partition $\overline{\Omega}$ of Ω. We construct a Cartesian decomposition $\overline{\mathcal{E}}$ of $\overline{\Omega}$ which is invariant under the action of $\overline{G}$ and has characteristics similar to those of $\mathcal{E}$. If $\mathcal{E} \in \mathsf{CD}_{2\sim}(G) \cup \mathsf{CD}_{1\rm S}(G)$ then a block in this partition will have size at most 2^k, where k depends on M, and often $k = 0$. This last statement will not, in general, be true for $\mathsf{CD}_{2\not\sim}(G)$.

[2] (Factorisation Property) We prove that T will admit some special type of factorisation. If $\mathcal{E} \in \mathsf{CD}_{2\sim}(G) \cup \mathsf{CD}_{1\mathrm{S}}(G)$ then this will enable us to severely restrict the isomorphism type of T (see the Isomorphism Property below). If $\mathcal{E} \in \mathsf{CD}_{2\not\sim}(G)$ then we will also exclude some isomorphism types for T.

[3] (Combinatorial Property) We prove that a point stabiliser G_ω preserves some combinatorial structure determined by G and $\mathcal{E}$, such as a graph or a generalised graph.

If $\mathcal{E} \in \mathsf{CD}_{2\sim}(G) \cup \mathsf{CD}_{1\mathrm{S}}(G)$ then we will also prove an isomorphism property.

[4] **(Isomorphism Property)** If $\mathcal{E} \in \mathsf{CD}_{2\sim}(G) \cup \mathsf{CD}_{1\mathrm{S}}(G)$ then the Factorisation Property is so strong that, up to an elementary abelian 2-group, we can pinpoint the permutational isomorphism type of the group G.

Theorems 5.1, 6.1 and 7.1 will be built on the above schema. The converse of these theorems will also hold in the following sense. If an innately transitive permutation group is given together with a factorisation and a combinatorial structure prescribed by the Factorisation Property and the Combinatorial Property, then we will show how to construct a Cartesian decomposition $\mathcal{E}$ that belongs to the corresponding class; see Sections 5.2, 6.2 and 7.3. For technical reasons in these constructions we will require that a point stabiliser in the plinth satisfies some extra condition to ensure that the partition in the Quotient Action Property will contain only trivial blocks. In particular, the constructions demonstrate that each instance where the Factorisation Property and the Combinatorial Property are satisfied can be realised by an innately transitive group G with Cartesian decomposition $\mathcal{E}$ of the appropriate type.

In Section 2 we collect necessary background information on Cartesian decompositions and Cartesian systems following the treatment in [BPS04], and [BPS06]. Section 3 contains some easy lemmas that we need for our main theorems. For an innately transitive group G, the Cartesian decompositions in $\mathcal{E} \in \mathsf{CD}_{2\sim}(G) \cup \mathsf{CD}_{2\not\sim}(G) \cup \mathsf{CD}_{1\mathrm{S}}(G)$ are studied via their quotient actions, and the required material is presented in Section 4. Sections 5, 6 and 7 are devoted to the classes $\mathsf{CD}_{2\sim}(G)$, $\mathsf{CD}_{2\not\sim}(G)$ and $\mathsf{CD}_{1\mathrm{S}}(G)$, respectively. We state and prove our main theorems in these three sections.

In this paper we use the following notation. Permutations act on the right: if π is a permutation and ω is a point then the image of ω under π is denoted ω^π. If G is a group acting on a set Ω then G^Ω denotes the subgroup of $\mathsf{Sym}\,\Omega$ induced by G. A transitive minimal normal subgroup of a permutation group is called a **plinth**; see [BamP04]. Note that a permutation group with a plinth is innately transitive, and that a finite innately transitive group can have at most two plinths, and if it has two, then they are isomorphic, non-abelian, and regular.

ACKNOWLEDGEMENT: The authors acknowledge the support of the Australian Research Council Discovery Grant DP0557587. Much of the work leading to this article was carried out while the second author was employed as a Research Associate in the Department of Mathematics and Statistics of The University of Western Australia; he was also supported by the Hungarian Scientific Research Fund (OTKA) Grant F049040.

2. Cartesian decompositions and Cartesian systems

A **Cartesian decomposition** of a set Ω is a set $\{\Gamma_1, \dots, \Gamma_\ell\}$ of proper partitions of Ω such that

$$|\gamma_1 \cap \dots \cap \gamma_\ell| = 1 \quad \text{for all } \gamma_1 \in \Gamma_1, \dots, \gamma_\ell \in \Gamma_\ell.$$

This property implies that the map $\omega \mapsto (\gamma_1, \dots, \gamma_\ell)$, where for $i = 1, \dots, \ell$ the block $\gamma_i \in \Gamma_i$ is chosen so that $\omega \in \gamma_i$, is a well-defined bijection between Ω and $\Gamma_1 \times \dots \times \Gamma_\ell$. Thus the set Ω can naturally be identified with the Cartesian product $\Gamma_1 \times \dots \times \Gamma_\ell$.

If G is a permutation group acting on Ω, then a Cartesian decomposition $\mathcal{E}$ of Ω is said to be G-invariant, if the partitions in $\mathcal{E}$ are permuted by G. For a permutation group G acting on Ω, the symbol $\mathsf{CD}(G)$ denotes the set of G-invariant Cartesian decompositions of Ω. If $\mathcal{E} \in \mathsf{CD}(G)$ and G acts on $\mathcal{E}$ transitively, then $\mathcal{E}$ is said to be a **transitive** G-invariant Cartesian decomposition. The set of transitive G-invariant Cartesian decompositions of Ω is denoted by $\mathsf{CD}_{\rm tr}(G)$. The concept of a Cartesian decomposition was introduced by L. G. Kovács in [Kov89b] where it is called a system of product imprimitivity. Kovács suggested that studying $\mathsf{CD}_{\rm tr}(G)$ (using our terminology), for finite primitive permutation groups G, was the appropriate way to identify inclusions of G in wreath products in their product action. His papers [Kob89b] and [Kov89a] inspired our work.

Suppose that G is an innately transitive subgroup of $\mathsf{Sym}\,\Omega$ with plinth M, and that $\mathcal{E}$ is a G-invariant Cartesian decomposition of Ω. Then we proved in [Proposition 2.1, BPS04] that each of the Γ_i is an M-invariant partition of Ω. Choose an element ω of Ω and let $\gamma_1 \in \Gamma_1, \dots, \gamma_\ell \in \Gamma_\ell$ be such that $\{\omega\} = \gamma_1 \cap \dots \cap \gamma_\ell$; set $K_i = M_{\gamma_i}$. Then [Lemmas 2.2 and 2.3, BPS04] imply that the set $\mathcal{K}_\omega(\mathcal{E}) = \{K_1, \ldots, K_\ell\}$ is invariant under conjugation by G_ω,

$$\bigcap_{i=1}^{\ell} K_i = M_\omega \tag{1}$$

and

$$K_i\left(\bigcap_{j\neq i} K_j\right) = M \quad \text{for all } i \in \{1, \ldots, \ell\}. \tag{2}$$

For an arbitrary transitive permutation group M on Ω and a point $\omega \in \Omega$, a set $\mathcal{K} = \{K_1, \ldots, K_\ell\}$ of proper subgroups of M is said to be a **Cartesian system of subgroups with respect to** ω for M, if (1) and (2) hold. If M is an abstract group then a set $\{K_1, \ldots, K_\ell\}$ of proper subgroups satisfying (2) is said to be a **Cartesian system**.

THEOREM 2.1 (Theorem 1.4 and Lemma 2.3, [BPS04]): *Let $G \leqslant \mathsf{Sym}\,\Omega$ be an innately transitive permutation group with plinth M. For a fixed $\omega \in \Omega$ the correspondence $\mathcal{E} \mapsto \mathcal{K}_\omega(\mathcal{E})$ is a bijection between the set of G-invariant Cartesian decompositions of Ω and the set of G_ω-invariant Cartesian systems of subgroups of M with respect to ω. Moreover the G_ω-actions on $\mathcal{E}$ and on $\mathcal{K}_\omega(\mathcal{E})$ are equivalent.*

Suppose that $G \leqslant \mathsf{Sym}\,\Omega$ is an innately transitive group with plinth M, and let $\omega \in \Omega$ be fixed. Let $\mathcal{K}$ be a G_ω-invariant Cartesian system of subgroups of M with respect to ω. Then the previous theorem implies that $\mathcal{K} = \mathcal{K}_\omega(\mathcal{E})$ for some G-invariant Cartesian decomposition $\mathcal{E}$ of Ω. Moreover, $\mathcal{E}$ consists of the M-invariant partitions $\{(\omega^K)^m \mid m \in M\}$ where K runs through the elements of $\mathcal{K}$. This Cartesian decomposition is usually denoted $\mathcal{E}(\mathcal{K})$.

Using this theory we were able to describe those innately transitive subgroups of wreath products that have a simple plinth. This led to a classification of transitive simple and almost simple subgroups of wreath products in product action (see [Theorem 1.1, BPS04]).

Now we recall a couple of concepts introduced in [BPS06] to describe subgroups of direct products. Suppose that $M = T_1 \times \cdots \times T_k$ where the T_i are groups, and $k \geqslant 1$. For $I \subseteq \{T_1, \ldots, T_k\}$ the symbol $\sigma_I\colon M \to \prod_{T_i \in I} T_i$ denotes the natural projection map. We also write $\sigma_{\{i_1,\ldots,i_m\}}$ for $\sigma_{\{T_{i_1},\ldots,T_{i_m}\}}$ and σ_i for $\sigma_{\{T_i\}}$. A subgroup X of M is said to be a **strip** if, for each $i = 1, \ldots, k$, either $\sigma_i(X) = 1$ or $\sigma_i(X) \cong X$. The set of T_i such that $\sigma_i(X) \neq 1$ is called the **support** of X and is denoted $\operatorname{Supp} X$. If $T_m \in \operatorname{Supp} X$ then we also say that X **covers** T_m. Two strips X_1 and X_2 are said to be **disjoint** if

$\operatorname{Supp} X_1 \cap \operatorname{Supp} X_2 = \emptyset$. A strip X is said to be **full** if $\sigma_i(X) = T_i$ for all $T_i \in \operatorname{Supp} X$, and it is called **non-trivial** if $|\operatorname{Supp} X| \geqslant 2$. A subgroup K of M is said to be **subdirect with respect to the direct decomposition** $T_1 \times \cdots \times T_k$ if $\sigma_i(K) = T_i$ for all i. If M is a finite non-abelian characteristically simple group, then a subgroup K is said to be **subdirect** if it is subdirect with respect to the finest direct decomposition of M (that is, as a direct product of simple groups).

Let $M = T_1 \times \cdots \times T_k$ be a finite non-abelian characteristically simple group, where $T_1, \ldots, T_k$ are the simple normal subgroups of M, each isomorphic to the same simple group T. If K is a subgroup of M and X is a strip in M such that $K = X \times \sigma_{\{T_1,\ldots,T_k\}\setminus \operatorname{Supp} X}(K)$ then we say that X is **involved** in K. A strip X is said to be involved in a Cartesian system $\mathcal{K}$ for M if X is involved in some element of $\mathcal{K}$. Note that in this case [Lemma 2.2, BP03] and (2) imply that X is involved in a unique element of $\mathcal{K}$.

A non-abelian plinth of an innately transitive group G has the form $M = T_1 \times \cdots \times T_k$ where the T_i are finite, non-abelian, simple groups. Let $\mathcal{E} \in \mathsf{CD}(G)$ and let $\mathcal{K}_\omega(\mathcal{E})$ be a corresponding Cartesian system $\{K_1, \ldots, K_\ell\}$ for M. Then equation (2) implies that, for all $i \leqslant k$ and $j \leqslant \ell$,

$$\sigma_i(K_j)\left(\bigcap_{j' \neq j} \sigma_i(K_{j'})\right) = T_i. \tag{3}$$

In particular this means that if $\sigma_i(K_j)$ is a proper subgroup of T_i then $\sigma_i(K_{j'}) \neq \sigma_i(K_j)$ for all $j' \in \{1, \ldots, \ell\} \setminus \{j\}$. It is thus important to understand the following sets of subgroups:

$$\mathcal{F}_i(\mathcal{E}, M, \omega) = \{\sigma_i(K_j) |\ j = 1, \ldots, \ell,\ \sigma_i(K_j) \neq T_i\}. \tag{4}$$

From our remarks above, $|\mathcal{F}_i(\mathcal{E}, M, \omega)|$ is the number of indices j such that $\sigma_i(K_j) \neq T_i$. The set $\mathcal{F}_i(\mathcal{E}, M, \omega)$ is independent of i up to isomorphism, in the sense that if $i_1, i_2 \in \{1, \ldots, k\}$ and $g \in G_\omega$ are such that $T_{i_1}^g = T_{i_2}$ then $\mathcal{F}_{i_1}(\mathcal{E}, M, \omega)^g = \{L^g |\ L \in \mathcal{F}_{i_1}(\mathcal{E}, M, \omega)\} = \mathcal{F}_{i_2}(\mathcal{E}, M, \omega)$. This argument also shows that the subgroups in $\mathcal{F}_{i_1}(\mathcal{E}, M, \omega)$ are actually G_ω-conjugate to the subgroups in $\mathcal{F}_{i_2}(\mathcal{E}, M, \omega)$.

The following theorem was proved in [Theorems 5.1 and 6.1, BPS06].

Theorem 2.2: *Suppose that G is an innately transitive permutation group with a non-abelian plinth $M = T_1 \times \cdots \times T_k$, where $k \geqslant 1$ and $T_1, \ldots, T_k$ are pairwise isomorphic finite simple groups. Let $\mathcal{E} \in \mathsf{CD}_{\mathrm{tr}}(G)$ with a corresponding*

Cartesian system $\mathcal{K}$ for M with respect to a point $\omega \in \Omega$, and, for $i = 1, \ldots, k$, let $\mathcal{F}_i = \mathcal{F}(\mathcal{E}, M, \omega)$ be defined as in (4). Then the following all hold.

(a) *The number $|\mathcal{F}_i|$ is independent of i and $|\mathcal{F}_i| \leqslant 3$.*

(b) *Suppose that there is a non-trivial, full strip involved in $\mathcal{K}$. Then $k \geqslant 2$ and $|\mathcal{F}_i| \in \{0,\ 1\}$.*

(c) *If X is a non-trivial, full strip involved in $\mathcal{K}$ and $|\mathcal{F}_i| = 1$ then $|\operatorname{Supp} X| = 2$.*

(d) *Set $\mathcal{P} = \{\operatorname{Supp} X \mid X$ is a non-trivial, full strip involved in $\mathcal{K}\}$. If $\mathcal{P} \neq \emptyset$ then $\mathcal{P}$ is a G-invariant partition of $\{T_1, \ldots, T_k\}$. Thus if X_1 and X_2 are non-trivial, full strips involved in $\mathcal{K}$ then they are disjoint.*

The set $\mathsf{CD}_{\mathrm{tr}}(G)$ is further subdivided according to the structure of the subgroups in the corresponding Cartesian systems as follows. The sets $\mathcal{F}_i = \mathcal{F}_i(\mathcal{E}, M, \omega)$ are defined as in (4).

$$\mathsf{CD}_{\mathrm{S}}(G) = \left\{\mathcal{E} \in \mathsf{CD}_{\mathrm{tr}}(G) \middle| \begin{array}{l}\text{the elements of } \mathcal{K}_\omega(\mathcal{E}) \text{ are subdirect subgroups}\\ \text{in } M\end{array}\right\};$$

$$\mathsf{CD}_1(G) = \left\{\mathcal{E} \in \mathsf{CD}_{\mathrm{tr}}(G)| \middle| \begin{array}{l}\mathcal{F}_i| = 1 \text{ and } \mathcal{K}_\omega(\mathcal{E}) \text{ involves no non-trivial,}\\ \text{full strip}\end{array}\right\};$$

$$\mathsf{CD}_{1\mathrm{S}}(G) = \left\{\mathcal{E} \in \mathsf{CD}_{\mathrm{tr}}(G) \middle| \begin{array}{l}|\mathcal{F}_i| = 1 \text{ and } \mathcal{K}_\omega(\mathcal{E}) \text{ involves non-trivial,}\\ \text{full strips}\end{array}\right\};$$

$$\mathsf{CD}_{2\sim}(G) = \left\{\mathcal{E} \in \mathsf{CD}_{\mathrm{tr}}(G) \middle| \begin{array}{l}|\mathcal{F}_i| = 2 \text{ and the } \mathcal{F}_i \text{ contain two } G_\omega\text{-conjugate}\\ \text{subgroups}\end{array}\right\};$$

$$\mathsf{CD}_{2\not\sim}(G) = \left\{\mathcal{E} \in \mathsf{CD}_{\mathrm{tr}}(G) \middle| \begin{array}{l}|\mathcal{F}_i| = 2 \text{ and the subgroups in } \mathcal{F}_i \text{ are not}\\ G_\omega\text{-conjugate}\end{array}\right\};$$

$$\mathsf{CD}_3(G) = \{\mathcal{E} \in \mathsf{CD}_{\mathrm{tr}}(G) \mid |\mathcal{F}_i| = 3\}.$$

At first glance, it seems that the definitions of the classes $\mathsf{CD}_{\mathrm{S}}(G)$, $\mathsf{CD}_1(G)$, $\mathsf{CD}_{1\mathrm{S}}(G)$, $\mathsf{CD}_{2\sim}(G)$, $\mathsf{CD}_{2\not\sim}(G)$ and $\mathsf{CD}_3(G)$ may depend on the choice of the Cartesian system, and hence on the choice of the point ω. The following result, proved in [Theorems 6.2 and 6.3, BPS06], shows that this is not the case, and shows also that these classes form a partition of $\mathsf{CD}_{\mathrm{tr}}(G)$. A permutation group is called **quasiprimitive** if all of its non-trivial normal subgroups are transitive. A finite quasiprimitive group is said to have **compound diagonal type** if it has a unique minimal normal subgroup M, which is non-abelian, and a point stabiliser M_ω is a non-simple, subdirect subgroup of M. See [BP03, Pra93] for more details.

Theorem 2.3 (6-class Theorem): *If G is a finite, innately transitive permutation group with a non-abelian plinth M, then the classes $\mathsf{CD}_1(G)$, $\mathsf{CD}_{\mathrm{S}}(G)$,*

$\mathsf{CD}_{1\mathrm{S}}(G)$, $\mathsf{CD}_{2\sim}(G)$, $\mathsf{CD}_{2\not\sim}(G)$ and $\mathsf{CD}_3(G)$ are independent of the choice of the point ω used in their definition. They form a partition of $\mathsf{CD}_{\mathrm{tr}}(G)$, and moreover, if M is simple, then $\mathsf{CD}_{\mathrm{tr}}(G) = \mathsf{CD}_{2\sim}(G)$.

(a) *If $\mathsf{CD}_{\mathrm{S}}(G) \neq \emptyset$, then G is a quasiprimitive group of compound diagonal type.*

(b) *If $\mathsf{CD}_{1\mathrm{S}}(G) \cup \mathsf{CD}_{2\sim}(G) \neq \emptyset$, then T and the subgroups of the $\mathcal{F}_i$ are given by one of the columns of Table 1.*

(c) *If $\mathsf{CD}_{2\not\sim}(G) \neq \emptyset$, then T admits a factorisation $T = AB$ with A, B proper subgroups.*

(d) *If $\mathsf{CD}_3(G) \neq \emptyset$, then T is isomorphic to one of the groups $\mathsf{Sp}_{4a}(2)$ with $a \geqslant 2$, $\mathsf{P}\Omega_8^+(3)$, or $\mathsf{Sp}_6(2)$, and, for each i, the subgroups of $\mathcal{F}_i$ form a strong multiple factorisation of T_i (see [Table V, BP98]), and hence are known explicitly.*

T	A_6	M_{12}	$\mathsf{P}\Omega_8^+(q)$	$\mathsf{Sp}_4(2^a),\ a \geqslant 2$
subgroups in $\mathcal{F}_i$	A_5	M_{11}	$\Omega_7(q)$	$\mathsf{Sp}_2(2^{2a}) \cdot 2$

Figure 1. Table for Theorem 2.3

3. Toolbox

In this section we collect the tools, in addition to those in [BPS04, BPS06, PS03], that are necessary for our investigation of the Cartesian decompositions in $\mathsf{CD}_{1\mathrm{S}}(G)$, $\mathsf{CD}_{2\sim}(G)$ and $\mathsf{CD}_{2\not\sim}(G)$. First we recall a couple of concepts in graph theory, and then we prove some group theoretic lemmas.

We introduce the combinatorial structures that are necessary for the investigation of the elements in $\mathsf{CD}_{2\sim}(G) \cup \mathsf{CD}_{2\not\sim}(G)$.

Definition 3.1: A **generalised di-graph** Γ is a 4-tuple $(V, E, \beta, \varepsilon)$, where V and E are disjoint sets with V non-empty and $\beta, \varepsilon \colon E \to V$ are maps such that $\beta(v) \neq \varepsilon(v)$ for all $v \in V$. The elements of V are the **vertices**, and the elements of E are the **arcs** of Γ. If $e \in E$ then $\beta(e)$ is the **initial vertex** of e, and $\varepsilon(e)$ is the **terminal vertex** of e. A permutation $\alpha \in \mathsf{Sym}\, V \times \mathsf{Sym}\, E \leqslant \mathsf{Sym}\,(V \cup E)$ is an **automorphism** of Γ if $\alpha(\beta(e)) = \beta(\alpha(e))$ and $\varepsilon(\alpha(e)) = \alpha(\varepsilon(e))$ for all $e \in E$.

Next we introduce the undirected version of this concept.

Definition 3.2: A **generalised graph** Γ is a triple (V, E, ε) where V and E are disjoint sets with V non-empty and

$$\varepsilon \colon E \to V^{\{2\}} = \{\{v_1, v_2\} \mid v_1, v_2 \in V,\ v_1 \neq v_2\}$$

is a map. The elements of V are the **vertices** and the elements of E are the **edges** of Γ. If $e \in E$ then the two elements of $\varepsilon(e)$ are said to be **adjacent** to e. A permutation $\alpha \in \mathsf{Sym}\, V \times \mathsf{Sym}\, E \leqslant \mathsf{Sym}\,(V \cup E)$ is an **automorphism** of Γ if $\varepsilon(\alpha(e)) = \alpha(\varepsilon(e))$ for all $e \in E$.

For the purposes of this paper, a **graph** is a generalised graph (V, E, ε) for which $E \subseteq V^{\{2\}}$ and ε is the inclusion map. We usually write this graph simply as (V, E), and with the terminology above, an edge $e = \{v_1, v_2\}$ is adjacent to v_1 and v_2 (and vice versa). We will also say that v_1 and v_2 are **connected**. A graph (V, E) is said to be **bipartite** if V has two non-empty subsets $V_1,\ V_2$ such that $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V$, and there is no edge between two elements of V_1 or between two elements of V_2. The pair $V_1,\ V_2$ is said to be a **bipartition** of the graph.

If $\Gamma = (V, E)$ is a graph then the **valency** of a vertex v is defined as the number $|\{v' \in V|\ \{v, v'\} \in E\}|$. A graph is said to be **regular** if all vertices have the same valency. A bipartite graph with a given bipartition is said to be **semiregular** if all vertices in the same part of the bipartition have the same valency.

A generalised graph can (and will) be viewed as a graph if there is at most one edge between any two vertices. For $n \geqslant 1$, the **complete graph** K_n is defined as the graph in which there are n vertices and any two vertices are connected.

Now we prove some lemmas that are necessary for our investigation.

LEMMA 3.3: *Let $\Gamma = (V, E, \varepsilon)$ be a generalised graph such that E is non-empty and $\mathsf{Aut}\,\Gamma$ induces a 2-transitive group on E. Then either $|V| = 2$ or Γ is a graph. In addition, if Γ is a graph and $\mathsf{Aut}\,\Gamma$ induces a transitive group on V, then Γ is isomorphic to the complete graph K_3, or Γ is isomorphic to a vertex-disjoint union of k copies of the complete graph K_2, for some $k \geqslant 1$.*

Proof: Since E is non-empty, we have $|V| \geqslant 2$. Note that $\mathsf{Aut}\,\Gamma$ must induce a primitive group on E. Suppose that v_1 and v_2 are vertices of Γ such that v_1 and v_2 are connected by some edge in E. Then the edges in E that are adjacent to v_1 and v_2 form a block for the action of $\mathsf{Aut}\,\Gamma$ on E. Thus either $|V| = 2$, or v_1 and v_2 are connected by a unique edge in E, and so Γ is a graph.

Assume now that Γ is a graph, and, in particular, that ε is an inclusion map, and that $\mathsf{Aut}\,\Gamma$ is transitive on V. This implies that all vertices have the same valency. If this valency is 1 then $\Gamma \cong kK_2$ for some k. So assume that the valency is at least 2 and let $v \in V$. Then there are edges e_1 and e_2 such that $e_1 = \{v, v_1\}$ and $e_2 = \{v, v_2\}$ with $v_1 \neq v_2$; in particular $|V| \geqslant 3$. As v_2 has valency at least 2, v_2 is adjacent to an edge $e_3 = \{v_2, v_3\}$ with $v_3 \neq v$. Since $\mathsf{Aut}\,\Gamma$ is 2-transitive on E, there is an automorphism $\alpha \in \mathsf{Aut}\,\Gamma$ such that $e_1^\alpha = e_1$ and $e_2^\alpha = e_3$. Thus

$$\{v\}^\alpha = (e_1 \cap e_2)^\alpha = e_1^\alpha \cap e_2^\alpha = e_1 \cap e_3 = \{v, v_1\} \cap \{v_2, v_3\}.$$

Since $v \notin \{v_2, v_3\}$ and $v_1 \neq v_2$, it follows that $v^\alpha = v_1 = v_3$. Thus the subgraph spanned by $v,\ v_1,\ v_2$ is a connected component of Γ and is a complete graph K_3. If $|V| \geqslant 4$, there is a vertex $v_4 \notin \{v, v_1, v_2\}$, and as $\mathsf{Aut}\,\Gamma$ is transitive on V, the connected component of Γ containing v_4 is also isomorphic to K_3. Let e be an edge adjacent to v_4. Since $\mathsf{Aut}\,\Gamma$ is 2-transitive on E, there is an automorphism β such that $(e_1, e_2)^\beta = (e_1, e)$. Arguing as before, $\{v^\beta\} = e_1 \cap e = \{v, v_1\} \cap e$, but this is the empty set, and we have a contradiction. Thus $|V| = 3$ and $\Gamma \cong K_3$. ∎

The next result, which will often be used in complicated arguments, is so easy that its proof is omitted.

Lemma 3.4: *Let A and B be subgroups of a group G, such that $A \lhd B$ and $\mathbb{N}_G\,(A)/A$ is abelian. Then $\mathbb{N}_G\,(A) \leqslant \mathbb{N}_G\,(B)$.*

The following result computes the normaliser of a strip in a direct product.

Lemma 3.5: *Let $G_1, \ldots, G_k$ be isomorphic groups, $\varphi_i\colon G_1 \to G_i$ an isomorphism for $i = 2, \ldots, k$, H_1 a subgroup of G_1, and*

$$H = \{(h, \varphi_2(h), \ldots, \varphi_k(h)) |\ h \in H_1\}$$

a non-trivial strip in $G_1 \times \cdots \times G_k$. Then

$$\begin{aligned} &\mathbb{N}_{G_1 \times \cdots \times G_k}\,(H) \\ &\qquad = \{(t, c_2\varphi_2(t), \ldots, c_k\varphi_k(t)) |\ t \in \mathbb{N}_{G_1}\,(H_1)\,,\ c_i \in \mathbb{C}_{G_i}\,(\varphi_i(H_1))\}. \end{aligned} \tag{5}$$

Proof: Denote the right hand side of equation (5) by N, set $G = G_1 \times \cdots \times G_k$, and consider an element $(t, c_2\varphi_2(t), \ldots, c_k\varphi_k(t)) \in N$. Then, for all $h \in H_1$,

$$(h, \varphi_2(h), \ldots, \varphi_k(h))^{(t, c_2\varphi_2(t), \ldots, c_k\varphi_k(t))} = (h^t, \varphi_2(h^t), \ldots, \varphi_k(h^t)) \in H.$$

Hence $N \subseteq \mathbb{N}_G\,(H)$. Let us prove that the other inclusion also holds. Suppose that $(t_1, \ldots, t_k) \in \mathbb{N}_G\,(H)$. Then for all $h \in H_1$ we have that

$$(h, \varphi_2(h), \ldots, \varphi_k(h))^{(t_1, \ldots, t_k)} = (h^{t_1}, \varphi_2(h)^{t_2}, \ldots, \varphi_k(h)^{t_k}) \in H,$$

and so $t_1 \in \mathbb{N}_{G_1}\,(H_1)$ and $\varphi_i(h^{t_1}) = \varphi_i(h)^{t_i}$ for each $i = 2, \ldots, k$. This amounts to saying, for each $i \geqslant 2$, that $h^{t_1} = h^{\varphi_i^{-1}(t_i)}$ for all $h \in H_1$, and hence

$t_1\varphi_i^{-1}(t_i)^{-1} \in \mathbb{C}_{G_1}(H_1)$. Therefore $t_i = c_i\varphi_i(t_1)$ for some $c_i \in \mathbb{C}_{G_i}(\varphi_i(H_1))$ for all $i = 2, \ldots, k$, and so

$$(t_1, \ldots, t_k) = (t_1, c_2\varphi_2(t_1), \ldots, c_k\varphi_k(t_k)).$$

Thus $\mathbb{N}_G(H) \subseteq N$, as required. ∎

Finally, we need one more result concerning factorisations of finite simple groups.

Lemma 3.6 (Lemma 4.2 [BPSxx]): *Let T be a finite simple group isomorphic to $\mathsf{Sp}_4(2^a)$, where $a \geqslant 2$, and let A, B be proper isomorphic subgroups of T such that $T = AB$. Then*

$$\mathbb{N}_T(A \cap B) = \mathbb{N}_T(A' \cap B') = A \cap B \quad \textit{and} \quad \mathbb{C}_T(A \cap B) = \mathbb{C}_T(A' \cap B') = 1.$$

4. Quotient actions of innately transitive groups

It is well-known that if H is a transitive permutation group on Ω then, for a fixed $\omega \in \Omega$, there is a one-to-one correspondence between the set $\{H_0 | \ H_\omega \leqslant H_0 \leqslant H\}$ of subgroups and the set of H-invariant partitions of Ω. The partition assigned to H_0 by this correspondence is denoted $\mathbb{P}_H(H_0)$, and is given by

$$\mathbb{P}_H(H_0) = \{(\omega^{H_0})^h | \ h \in H\}. \tag{6}$$

In particular, the part of $\mathbb{P}_H(H_0)$ containing ω is the H_0-orbit ω^{H_0}, and H_0 is its setwise stabiliser in H. Note that the next result does not assume that Ω is finite.

Lemma 4.1: *Let G be a permutation group on a set Ω and M a transitive normal subgroup of G. Suppose that for some $\omega \in \Omega$, $M_\omega \leqslant M_0 \leqslant M$ and M_0 is normalised by G_ω. Then $\mathbb{P}_M(M_0)$ is G-invariant, and if $P \in \mathbb{P}_M(M_0)$ such that $\omega \in P$ then $G_P = M_0G_\omega$. Moreover, $\mathbb{P}_M(M_0) = \mathbb{P}_G(M_0G_\omega)$.*

Proof: It is clear from its definition that $\mathbb{P}_M(M_0)$ is M-invariant. Since M is transitive, we have $G = MG_\omega$, and so in order to show that $\mathbb{P}_M(M_0)$ is G-invariant, it suffices to show that $\mathbb{P}_M(M_0)$ is G_ω-invariant. If $g \in G_\omega$ and $m \in M$ then

$$(\omega^{M_0m})^g = \omega^{M_0mg} = \omega^{M_0gm^g} = \omega^{gM_0m^g} = \omega^{M_0m^g} \in \mathbb{P}_M(M_0).$$

Hence $\mathbb{P}_M(M_0)$ is G-invariant. Thus, by the remarks preceding the lemma, $\mathbb{P}_M(M_0) = \mathbb{P}_G(X)$ for a unique subgroup X satisfying $G_\omega \leqslant X \leqslant G$, the part $P = \omega^{M_0}$ containing ω is the X-orbit ω^X, and X is its setwise stabiliser in G. Since $\mathbb{P}_M(M_0)$ is G-invariant, it follows that G_ω fixes P setwise, as does M_0, and by assumption $M_0 G_\omega = G_\omega M_0$ is a subgroup of G containing G_ω. Since $\omega^{G_\omega M_0} = \omega^{M_0} = \omega^X$, the uniqueness of X implies that $X = G_\omega M_0$. ∎

Lemma 4.1 can be used to construct quotient actions of innately transitive groups. Suppose that M is a non-abelian, transitive, minimal normal subgroup of a finite permutation group G, acting on Ω. Let $\omega \in \Omega$ and let $\mathcal{M}$ be a G-invariant partition of the minimal normal subgroups of M. If, for $I \in \mathcal{M}$, σ_I denotes the projection of M to the direct product of the minimal normal subgroups lying in I, then $M_\omega \leqslant \prod_{I\in\mathcal{M}} \sigma_I(M_\omega) \leqslant M$, and we define

$$\mathbb{P}(\mathcal{M}) = \mathbb{P}_M\left(\prod_{I\in\mathcal{M}} \sigma_I(M_\omega)\right).$$

As $\sigma_I(M_\omega)^g = \sigma_{I^g}(M_\omega)$ for all $I \in \mathcal{M}$ and $g \in G_\omega$, the subgroup $\prod_I \sigma_I(M_\omega)$ is normalised by G_ω. Therefore $\mathbb{P}(\mathcal{M})$ is an M-invariant partition of Ω.

5. Cartesian decompositions in $\mathsf{CD}_{2\sim}(G)$

In this section we assume that G is an innately transitive group acting on Ω with a non-abelian plinth $M = T_1 \times \cdots \times T_k$ where each of the T_i is isomorphic to a finite simple group T. Set $\mathcal{T} = \{T_1, \ldots, T_k\}$, and fix an $\omega \in \Omega$. Let $\overline{M}_\omega = \mathbb{N}_{T_1}(\sigma_1(M_\omega)) \times \cdots \times \mathbb{N}_{T_k}(\sigma_k(M_\omega))$ and let $\overline{\Omega}$ denote the M-invariant partition $\mathbb{P}_M\left(\overline{M}_\omega\right)$ of Ω. Using Lemma 4.1, it is easy to see that $\overline{\Omega}$ is G-invariant. Let $\overline{\omega}$ be the block in $\overline{\Omega}$ that contains ω. Then $\overline{M}_\omega = M_{\overline{\omega}}$ and Lemma 4.1 also implies that $G_{\overline{\omega}} = \overline{M}_\omega G_\omega$. Let $\overline{G}$ denote the group induced by G on $\overline{\Omega}$, and let $\overline{G}_{\overline{\omega}}$ denote the image in $\overline{G}$ of $G_{\overline{\omega}}$.

Suppose that $\mathcal{E} \in \mathsf{CD}_{2\sim}(G)$, and for each $i = 1, \ldots, k$, let $\mathcal{F}_i(\mathcal{E}, M, \omega) = \{A_i, B_i\}$. Let $\Gamma(G, \mathcal{E})$ be the generalised graph $(\mathcal{K}_\omega(\mathcal{E}), \mathcal{T}, \varepsilon)$ such that, for $i = 1, \ldots, k$, $\varepsilon(T_i) = \{K_{j_1},\ K_{j_2}\}$ where $\sigma_i(K_{j_1}) = A_i$ and $\sigma_i(K_{j_2}) = B_i$. For $i = 1, \ldots, \ell$, set $\overline{K}_i = \sigma_1(K_i) \times \cdots \times \sigma_k(K_i)$, and let $\overline{\mathcal{K}}_\omega(\mathcal{E}) = \{\overline{K}_1, \ldots, \overline{K}_\ell\}$.

The main result of this section is the following theorem.

Theorem 5.1: *Let the groups G and M be as in the first paragraph of this section. If $\mathcal{E} \in \mathsf{CD}_{2\sim}(G)$, then the properties* Prop2∼[a]–[d] *below all hold.*

Prop2∼[a] (Quotient Action Property). The group M is faithful on $\overline{\Omega}$, and so, if K is a subgroup of M, then we identify K with its image under the action on $\overline{\Omega}$. The set $\overline{\mathcal{K}}_\omega(\mathcal{E})$ is a $\overline{G}_{\overline{\omega}}$-invariant Cartesian system of subgroups for M with respect to $\overline{\omega}$. Moreover, $\mathcal{E}(\overline{\mathcal{K}}_\omega(\mathcal{E})) \in \mathsf{CD}_{2\sim}(\overline{G})$.

Prop2∼[b] (Factorisation Property). If $i \in \{1, \dots, k\}$ then

(i) $A_i,\ B_i$ are isomorphic proper subgroups of T_i;

(ii) A_i and B_i are conjugate under G_ω;

(iii) $A_iB_i = T_i$, $A_i \cap B_i = \mathbb{N}_{T_i}(\sigma_i(M_\omega))$;

(iv) $\mathbb{N}_{G_\omega}(T_i) = \{g \in G_\omega |\ \{A_i,\ B_i\}^g = \{A_i,\ B_i\}\}$.

Prop2∼[c] (Combinatorial Property). The group G_ω induces a group of automorphisms of the generalised graph $\Gamma(G, \mathcal{E})$, which is transitive on both the vertex-set $\mathcal{K}_\omega(\mathcal{E})$ and the edge-set $\mathcal{T}$, where the G_ω-actions are by conjugation. Moreover, if for some $i \in \{1, \dots, k\}$, $\varepsilon(T_i) = \{K_{j_1}, K_{j_2}\}$ and $g \in \mathbb{N}_{G_\omega}(T_i)$, then $(A_i, B_i)^g = (A_i, B_i)$ if and only if $(K_{j_1}, K_{j_2})^g = (K_{j_1}, K_{j_2})$.

Prop2∼[d] (Isomorphism Property). The group T, the subgroups of $\mathcal{F}_i(\mathcal{E}, M, \omega)$, and $\sigma_i(M_{\overline{\omega}})$ are as in Table 2. The group $\overline{G}$ is permutationally isomorphic to a subgroup of $\mathsf{Aut}\, M$ acting on $\overline{\Omega}$. In particular, M is the unique minimal normal subgroup of $\overline{G}$, and $\overline{G}$ is quasiprimitive. Moreover, if T is as in rows 1–3 of Table 2 then $\overline{M}_\omega = M_\omega$, and so $G \cong \overline{G}$, as permutation groups. Otherwise a block in $\overline{\Omega}$ has size dividing 2^k, the kernel N of the action of G on $\overline{\Omega}$ is an elementary abelian 2-group of rank at most k, and $\overline{G} \cong G/N$.

A converse of Theorem 5.1 is also true, see Section 5.2. The following proposition will form the basis for the proof of Theorem 5.1.

Proposition 5.2: *Suppose that G, M, T, ω, $\mathcal{E}$, $\mathcal{F}_i(\mathcal{E}, M, \omega)$ are as in the first and second paragraphs of this section. Then the isomorphism type of T and that of the subgroups in $\mathcal{F}_i(\mathcal{E}, M, \omega)$ are as in one of the rows of Table 2. If one of the rows 1–3 of Table 2 is valid then*

$$K = \sigma_1(K) \times \cdots \times \sigma_k(K) \quad \text{for } K \in \mathcal{K}_\omega(\mathcal{E}), \tag{7}$$

while if row 4 is valid then

$$\sigma_1(K)' \times \cdots \times \sigma_k(K)' \leqslant K \quad \text{and} \quad \frac{K}{\sigma_1(K)' \times \cdots \times \sigma_k(K)'} \leqslant \mathbb{Z}_2^k \quad \text{for } K \in \mathcal{K}_\omega(\mathcal{E}). \tag{8}$$

	T	subgroups in $\mathcal{F}_i(\mathcal{E}, M, \omega)$	$\sigma_i(M_{\overline{\omega}})$
1	A_6	A_5	D_{10}
2	M_{12}	M_{11}	$\mathsf{PSL}_2(11)$
3	$\mathsf{P\Omega}_8^+(q)$	$\Omega_7(q)$	$\mathsf{G}_2(q)$
4	$\mathsf{Sp}_4(q),\ q \geqslant 4$ even	$\mathsf{Sp}_2(q^2) \cdot 2$	$\mathsf{D}_{q^2+1} \cdot 2$

Table 2. Table for Proposition 5.2

Proof: For $i = 1, \ldots, k$, we have $\mathcal{F}_i(\mathcal{E}, M, \omega) = \{A_i, B_i\}$, as above. Since G acts transitively on $\mathcal{T}$ by conjugation, and since, by the definition of $\mathsf{CD}_{2\sim}(G)$, A_i and B_i are G_ω-conjugate, the subgroups $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ are pairwise isomorphic. Also, since $T_1 = A_1 B_1$ is a factorisation of a finite simple group with two isomorphic subgroups, it follows from [Lemma 5.2, BPS04] that T and $\mathcal{F}_1(\mathcal{E}, M, \omega)$ are as in Table 2. Suppose that $\sigma_1(K_j)' \times \cdots \times \sigma_k(K_j)' \not\leqslant K_j$, for some j. Then it follows from [Lemma 2.3, PS02] that there are $i_1,\ i_2 \in \{1, \ldots, k\}$ such that

$$\sigma_{i_1}(K_j)' \times \sigma_{i_2}(K_j)' \not\leqslant \sigma_{\{i_1,i_2\}}(K_j). \tag{9}$$

Suppose first that $\sigma_{i_1}(K_j) = T_{i_1}$. Then [Lemma 4.2, BPS06] implies that K_j involves a full strip X covering T_{i_1}. However, by Theorem 2.2, X cannot be a non-trivial strip since $\mathcal{E} \in \mathsf{CD}_{2\sim}(G)$. Thus $X = T_{i_1}$, and so $T_{i_1} \leqslant K_j$. This, however, implies that $\sigma_{i_1}(K_j) \leqslant \sigma_{\{i_1,i_2\}}(K_j)$, and in this case we must also have $\sigma_{i_2}(K_j) \leqslant \sigma_{\{i_1,i_2\}}(K_j)$. Therefore $\sigma_{i_1}(K_j) \times \sigma_{i_2}(K_j) = \sigma_{\{i_1,i_2\}}(K_j)$ contradicting (9). Hence $\sigma_{i_1}(K_j) < T_{i_1}$, and the same argument shows that $\sigma_{i_2}(K_j) < T_{i_2}$.

Since $\mathcal{E} \in \mathsf{CD}_{2\sim}(G)$, there exist $j_1, j_2 \in \{1, \ldots, \ell\} \setminus \{j\}$ such that $\sigma_{i_1}(K_{j_1}) < T_{i_1}$ and $\sigma_{i_2}(K_{j_2}) < T_{i_2}$. It follows from (2) that $K_j(K_{j_1} \cap K_{j_2}) = M$ (where possibly $j_1 = j_2$) and so

$$\sigma_{\{i_1,i_2\}}(K_j)(\sigma_{\{i_1,i_2\}}(K_{j_1}) \cap \sigma_{\{i_1,i_2\}}(K_{j_2})) = T_{i_1} \times T_{i_2}.$$

Note that

$$\sigma_{\{i_1,i_2\}}(K_{j_1}) \cap \sigma_{\{i_1,i_2\}}(K_{j_2}) \leqslant \sigma_{i_1}(K_{j_1}) \times \sigma_{i_2}(K_{j_2})$$

and hence

$$\sigma_{\{i_1,i_2\}}(K_j)(\sigma_{i_1}(K_{j_1}) \times \sigma_{i_2}(K_{j_2})) = T_{i_1} \times T_{i_2}.$$

By an observation made at the beginning of this proof, $\sigma_{i_1}(K_j)$, $\sigma_{i_2}(K_j)$, $\sigma_{i_1}(K_{j_1})$, $\sigma_{i_2}(K_{j_2})$ are pairwise isomorphic. Therefore the factorisation in the previous displayed equation is a full factorisation (see [Definition 1.1, PS02]).

On the other hand (9) holds, and this contradicts [Theorem 1.2, PS02]. Hence the first inequality of (8) holds for all $K \in \mathcal{K}_\omega(\mathcal{E})$. If T is not as in row 4 of Table 2 then the elements of the $\mathcal{F}_i$ are finite simple groups, and the stronger equation (7) also follows.

Finally if T is as in row 4 of Table 2 then $\sigma_i(K_j)/\sigma_i(K_j)' \cong \mathbb{Z}_2$, and hence

$$\frac{K_j}{\sigma_1(K_j)' \times \cdots \times \sigma_k(K_j)'} \leqslant \frac{\sigma_1(K_j) \times \cdots \times \sigma_k(K_j)}{\sigma_1(K_j)' \times \cdots \times \sigma_k(K_j)'} \cong \mathbb{Z}_2^k. \qquad \blacksquare$$

Now we prove Theorem 5.1.

5.1. Proof of Theorem 5.1. **Prop2∼[a]** As $\mathcal{E} \in \mathsf{CD}_{2\sim}(G)$ we have that, for all i, there are two indices j such that $\sigma_i(K_j) < T_i$. Hence each of the $\overline{K}_i$ is a proper subgroup of M. This also shows that $\sigma_i(M_\omega)$ is a proper subgroup of T_i, and so is $\mathbb{N}_{T_i}(\sigma_i(M_\omega))$, for all $i \in \{1,\dots,k\}$. Thus no T_i is contained in $\overline{M}_\omega$, and so M is faithful on $\overline{\Omega}$. We will therefore identify each subgroup K of M with its image under the action on $\overline{\Omega}$. Set $\mathcal{K} = \mathcal{K}_\omega(\mathcal{E})$ and $\overline{\mathcal{K}} = \overline{\mathcal{K}}_\omega(\mathcal{E})$.

Next we prove that $\overline{\mathcal{K}}$ is a $\overline{G_{\overline{\omega}}}$-invariant Cartesian system for M with respect to $\overline{\omega}$. If one of the rows 1–3 of Table 2 is valid, then, by Proposition 5.2, $\overline{\mathcal{K}} = \mathcal{K}$ and $\overline{\omega} = \{\omega\}$ and there is nothing to prove. Thus we suppose that row 4 of Table 2 is valid. First we prove that (1) holds. Let $i \in \{1,\dots,k\}$ and $\mathcal{F}_i = \{A_i, B_i\}$. Then it follows from (8) that $A_i' \cap B_i' \leqslant \sigma_i(M_\omega) \leqslant A_i \cap B_i$. As $A_iB_i = A_i'B_i = A_iB_i' = T_i$ but $A_i'B_i' \neq T_i$, we obtain that $|A_i \cap B_i : A_i' \cap B_i'| = 2$. Lemma 3.6 implies that $\mathbb{N}_{T_i}(A_i' \cap B_i') = \mathbb{N}_{T_i}(A_i \cap B_i) = A_i \cap B_i$. Hence $\mathbb{N}_{T_i}(\sigma_i(M_\omega)) = A_i \cap B_i$, and so

$$\overline{K}_1 \cap \cdots \cap \overline{K}_\ell = \prod_{i=1}^{k}(A_i \cap B_i) = \prod_{i=1}^{k} \mathbb{N}_{T_i}(\sigma_i(M_\omega)) = M_{\overline{\omega}}$$

and condition (1) is proved. Since (2) holds for $\mathcal{K}$ and $K_i \leqslant \overline{K}_i$ for all i, we obtain that(2) holds for $\overline{\mathcal{K}}$ as well.

We claim that $\overline{\mathcal{K}}$ is invariant under conjugation by $G_{\overline{\omega}}$. Let $i \in \{1,\dots,k\}$, $j \in \{1,\dots,\ell\}$ and $g \in G_\omega$. We denote by i^g the integer in $\{1,\dots,k\}$ that satisfies $T_i^g = T_{i^g}$. Then $\sigma_i(K_j)^g = \sigma_{i^g}(K_j^g)$. Thus

$$(\overline{K}_j)^g = \left(\prod_{i=1}^{k} \sigma_i(K_j)\right)^g = \prod_{i=1}^{k} \sigma_{i^g}(K_j^g) = \prod_{i=1}^{k} \sigma_i(K_j^g).$$

Since $K_j^g \in \mathcal{K}$, it follows that $\overline{K_j^g} \in \overline{\mathcal{K}}$. Hence $\overline{\mathcal{K}}$ is G_ω-invariant. Lemma 4.1 shows that $G_{\overline{\omega}} = \overline{M}_\omega G_\omega$. Since $\overline{M}_\omega = \overline{K}_1 \cap \cdots \cap \overline{K}_\ell$ preserves $\overline{\mathcal{K}}$, we obtain that $\overline{\mathcal{K}}$ is also $G_{\overline{\omega}}$-invariant. Thus $\overline{\mathcal{K}}$ is a $\overline{G}_{\overline{\omega}}$-invariant Cartesian system of subgroups for M with respect to $\overline{\omega}$.

It follows from the definition of $\overline{\mathcal{K}}$ that $\mathcal{F}_i(\mathcal{E}, M, \omega) = \mathcal{F}_i(\mathcal{E}(\overline{\mathcal{K}}), M, \overline{\omega}) = \{A_i, B_i\}$ for all $i \in \{1, \ldots, k\}$. Let $g \in G_\omega$ such that $A_1^g = B_1$ and let $\overline{g}$ denote the image of g in its action on $\overline{\Omega}$. Then $\overline{g} \in \overline{G}_{\overline{\omega}}$ and clearly $A_1^{\overline{g}} = B_1$. Thus $\mathcal{E}(\overline{\mathcal{K}}) \in \mathsf{CD}_{2\sim}(\overline{G})$.

Prop2∼[b] Let $i \in \{1, \ldots, k\}$ and choose $j_1, j_2 \in \{1, \ldots, \ell\}$ such that $A_i = \sigma_i(K_{j_1})$ and $B_i = \sigma_i(K_{j_2})$. It is clear by the definition of $\mathsf{CD}_{2\sim}(G)$ that Prop2∼[b](i)–(ii) hold for A_i and B_i. Since $K_{j_1}K_{j_2} = M$ we have that $\sigma_i(K_{j_1})\sigma_i(K_{j_2}) = \sigma_i(M)$ and so $A_iB_i = T_i$. We showed in the proof of Prop2∼[a] that $A_i \cap B_i = \mathbb{N}_{T_i}(\sigma_i(M_\omega))$, and so Prop2∼[b](iii) also holds. Finally, let $g \in G_\omega$ such that $\{A_i, B_i\}^g = \{A_i, B_i\}$. Since $A_i,\ B_i \leqslant T_i$, it follows that $T_i \cap T_i^g \neq 1$, and so $T_i^g = T_i$. Conversely, if $T_i^g = T_i$ with some $g \in G_\omega$, then $\sigma_i(K)^g = \sigma_i(K^g)$ for all $K \in \mathcal{K}$. Thus the uniqueness of $\{j_1, j_2\}$ yields that g fixes $\{K_{j_1}, K_{j_2}\}$. Therefore g fixes $\{\sigma_i(K_{j_1}), \sigma_i(K_{j_2})\} = \{A_i, B_i\}$. Thus all properties in Prop2∼[b] hold.

Prop2∼[c] Let Γ denote $\Gamma(G, \mathcal{E})$. It follows from the definition of Γ that the action of G_ω by conjugation is transitive on the vertex set $\mathcal{K}$ and on the edge set $\mathcal{T}$ of Γ. We claim that G_ω preserves adjacency in $\Gamma(G, \mathcal{E})$. Let $i_1 \in \{1, \ldots, k\}$ with $\varepsilon(T_{i_1}) = \{K_{j_1}, K_{j_2}\}$ and let $g \in G_\omega$. Let $i_2 \in \{1, \ldots, k\}$ such that $T_{i_1}^g = T_{i_2}$. Then $\sigma_{i_1}(K_{j_1})^g = \sigma_{i_2}(K_{j_1}^g)$ and $\sigma_{i_1}(K_{j_2})^g = \sigma_{i_2}(K_{j_2}^g)$. Thus $\varepsilon(T_{i_1}^g) = \varepsilon(T_{i_2}) = \{K_{j_1}^g, K_{j_2}^g\} = \varepsilon(T_{i_1})^g$, as required.

Suppose that $g \in \mathbb{N}_{G_\omega}(T_i)$. If g is such that $A_i^g = A_i$ and $B_i^g = B_i$ then, since $g \in \mathbb{N}_{G_\omega}(T_i)$, $A_i = A_i^g = \sigma_i(K_{j_1})^g = \sigma_i(K_{j_1}^g)$. Since j_1 is the unique integer in $\{1, \ldots, \ell\}$ such that $\sigma_i(K_{j_1}) = A_i$, we obtain that $K_{j_1}^g = K_{j_1}$, and also $K_{j_2}^g = K_{j_2}$. If $g \in \mathbb{N}_{G_\omega}(T_i)$ is such that $K_{j_1}^g = K_{j_1}$ and $K_{j_2}^g = K_{j_2}$ then it also follows that $A_i^g = \sigma_i(K_{j_1})^g = \sigma_i(K_{j_1}^g) = \sigma_i(K_{j_1}) = A_i$, and, of course, $B_i^g = B_i$. Thus the compatibility condition between A_i, B_i, and $\Gamma(G, \mathcal{E})$ in Prop2∼[c] also holds.

Prop2∼[d] It follows from Proposition 5.2 that T and the subgroups $A_i,\ B_i$ of $\mathcal{F}_i(\mathcal{E}, M, \omega)$ are as in Table 2. As $M_{\overline{\omega}} = \overline{K}_1 \cap \cdots \cap \overline{K}_\ell$, we obtain that $\sigma_i(M_{\overline{\omega}}) = A_i \cap B_i$ for all i. Since $T_i = A_iB_i$ is a factorisation of T_i with two isomorphic subgroups, we obtain from the [Atlas] in rows 1–2, from [3.1.1(vi), Kle87] in row 3, and from [3.2.1(d), LPS90] in row 4 of Table 2 that the $\sigma_i(M_{\overline{\omega}})$-column of Table 2 is correct. As $M_{\overline{\omega}} = \overline{M}_\omega$ is the direct product of its projections under the σ_i, and such a projection is self-normalising in T_i (by Lemma 3.6), we obtain that $M_{\overline{\omega}}$ is a self-normalising subgroup in M. Thus [Theorem 4.2A, DM96] implies that $\mathbb{C}_{\mathsf{Sym}\,\overline{\Omega}}(M) = 1$. Hence M is the unique minimal normal subgroup of $\overline{G}$, and so $\overline{G}$ can be embedded into a subgroup of $\mathsf{Aut}\,M$. In particular, $\overline{G}$ is quasiprimitive.

Suppose that

$$\underline{K}_i = \sigma_1(K_i)' \times \cdots \times \sigma_k(K_i)' \quad \text{for all } i \in \{1, \ldots, \ell\}$$

and set $\underline{M}_\omega = \underline{K}_1 \cap \cdots \cap \underline{K}_\ell$. It follows from Theorem 5.2 that $\underline{K}_i \leqslant K_i \leqslant \overline{K}_i$ and that $\underline{K}_i = K_i = \overline{K}_i$ if T is as in one of the rows 1–3 of Table 2; thus $\underline{M}_\omega \leqslant M_\omega \leqslant M_{\overline{\omega}}$ also holds. If T is as in one of the rows 1–3 of Table 2, then $\underline{M}_\omega = M_\omega = M_{\overline{\omega}}$. Thus $\overline{\Omega}$ can be identified with Ω, and so the groups G and $\overline{G}$ are permutationally isomorphic.

Suppose now that T is as in row 4 of Table 2. Then, by [3.2.1(d), LPS90], $\sigma_i(M_{\overline{\omega}}) \cong A_i \cap B_i \cong \mathsf{D}_{q^2+1} \cdot 2$ and $\sigma_i(\underline{M}_\omega) \cong A_i' \cap B_i' \cong \mathsf{D}_{q^2+1}$ for all $i \in \{1, \ldots, k\}$. It follows from Lemma 3.6 that

$$\mathbb{N}_{T_i}\left(\sigma_i(\underline{M}_\omega)\right) = \mathbb{N}_{T_i}\left(\sigma_i(M_{\overline{\omega}})\right) = \sigma_i(M_{\overline{\omega}}) \quad \text{for all } i \in \{1, \ldots, k\}.$$

Hence we obtain that $\mathbb{N}_M(M_\omega) \leqslant \mathbb{N}_M(\underline{M}_\omega) = M_{\overline{\omega}}$. On the other hand, as $\mathbb{N}_M(\underline{M}_\omega)/\underline{M}_\omega$ is abelian, Lemma 3.4 gives $\mathbb{N}_M(\underline{M}_\omega) \leqslant \mathbb{N}_M(M_\omega)$. Thus $\mathbb{N}_M(\underline{M}_\omega) = \mathbb{N}_M(M_\omega)$. Therefore $\mathbb{N}_M(M_\omega)/M_\omega$ is an elementary abelian 2-group of rank at most k, and by [Theorem 4.2A, DM96] a block in $\overline{\Omega}$ also has size dividing 2^k. Therefore N is also an elementary abelian 2-group of rank at most k. ∎

5.2. A converse of Theorem 5.1. Theorem 5.1 can be reversed in the following sense. Let G be a finite innately transitive group on Ω with a non-abelian plinth M and let $T_1, \ldots, T_k$ be the simple direct factors of M. Assume that, for $\omega \in \Omega$, the point stabiliser M_ω can be decomposed as $M_\omega = \sigma_1(M_\omega) \times \cdots \times \sigma_k(M_\omega)$. Set $\mathcal{T} = \{T_1, \ldots, T_k\}$. Suppose that A_1, B_1 are subgroups of T_1 and $\Gamma = (V, \mathcal{T}, \varepsilon)$ is a generalised graph, such that properties Prop2∼[b] and Prop2∼[c] hold. More precisely,

(i) A_1, B_1 are isomorphic proper subgroups of T_1;
(ii) A_1 and B_1 are conjugate under G_ω;
(iii) $A_1 B_1 = T_1$, $A_1 \cap B_1 = \sigma_1(M_\omega)$;
(iv) $\mathbb{N}_{G_\omega}(T_1) = \{g \in G_\omega \mid \{A_1,\ B_1\}^g = \{A_1,\ B_1\}\}$.

Assume, moreover, that G_ω induces a vertex and edge-transitive group of automorphisms of Γ, where the G_ω-action on $\mathcal{T}$ is by conjugation, and that if

$\varepsilon(T_1) = \{v_1, v_2\}$ in Γ, then the following holds:

$$\text{(10)}\qquad \text{if } g \in \mathbb{N}_{G_\omega}(T_1) \text{ then } (A_1, B_1)^g = (A_1, B_1) \quad \text{if and only if } (v_1, v_2)^g = (v_1, v_2).$$

We construct, as follows, a G-invariant Cartesian decomposition $\mathcal{E}$ in $\mathsf{CD}_{2\sim}(G)$, such that $\Gamma \cong \Gamma(G, \mathcal{E})$ and $\mathcal{F}_1(\mathcal{E}, M, \omega) = \{A_1, B_1\}$.

For $i = 1, \ldots, k$, choose $g_i \in G_\omega$ such that $T_1^{g_i} = T_i$. For each element $v \in V$ set $K_v = \prod_{i=1}^k K_{v,i}$ where, for $i = 1, \ldots, k$, the subgroup $K_{v,i}$ is defined as follows (noting that $\varepsilon(T_i) = \{v_1^{g_i}, v_2^{g_i}\}$). Set $K_{v_1^{g_i},i} = A_1^{g_i}$, $K_{v_2^{g_i},i} = B_1^{g_i}$ and $K_{v,i} = T_i$ for all $v \in V \setminus \{v_1^{g_i}, v_2^{g_i}\}$.

We claim that each of the $K_{v,i}$ is well-defined, that is, its definition is independent of the choice of the g_i. Suppose that g_i, $g_i' \in G_\omega$ are such that $T_1^{g_i} = T_1^{g_i'} = T_i$ for some i. Note that, as G_ω induces a group of automorphisms of Γ, in this case $\{v_1^{g_i}, v_2^{g_i}\} = \varepsilon(T_i) = \{v_1^{g_i'}, v_2^{g_i'}\}$. Thus if $v \notin \{v_1^{g_i}, v_2^{g_i}\}$ then we would define $K_{v,i}$ as T_i using either g_i or g_i'. Suppose next that $v_1^{g_i} = v_1^{g_i'}$ and $v_2^{g_i} = v_2^{g_i'}$. Then $g_i g_i'^{-1} \in \mathbb{N}_{G_\omega}(T_1) \cap (G_\omega)_{v_1}$ and so, by (10), $g_i g_i'^{-1} \in \mathbb{N}_{G_\omega}(A_1) \cap \mathbb{N}_{G_\omega}(B_1)$. Thus $A_1^{g_i g_i'^{-1}} = A_1$ and $B_1^{g_i g_i'^{-1}} = B_1$; and so $A_1^{g_i} = A_1^{g_i'}$ and $B_1^{g_i} = B_1^{g_i'}$. Therefore, using either g_i or g_i', we would define $K_{v_1^{g_i},i}$ as $A_1^{g_i}$ and $K_{v_2^{g_i},i}$ as $B_1^{g_i}$. The other possibility is that $v_1^{g_i} = v_2^{g_i'}$ and $v_1^{g_i'} = v_2^{g_i}$. Then $g_i g_i'^{-1}$ is in $\mathbb{N}_{G_\omega}(T_1)$ and interchanges v_1 and v_2, and so by property (iv) above and condition (10), $g_i g_i'^{-1}$ also swaps A_1 and B_1. For $v = v_1^{g_i} = v_2^{g_i'}$ we would, using g_i', define $K_{v,i}$ as $B_1^{g_i'} = (A_1^{g_i g_i'^{-1}})^{g_i'} = A_1^{g_i}$, and similarly, for $v = v_2^{g_i} = v_1^{g_i'}$ we would, using g_i', define $K_{v,i}$ as $A_1^{g_i'} = (B_1^{g_i g_i'^{-1}})^{g_i'} = B_1^{g_i}$. Thus the definitions of all the $K_{v,i}$ are the same whether we use g_i or g_i'.

Let $\mathcal{K} = \{K_v |\ v \in V\}$. We claim that $\mathcal{K}$ is a G_ω-invariant Cartesian system for M with respect to ω. First note that the K_v are direct products of their projections and, for all i,

$$\bigcap_{v \in V} K_{v,i} = A_1^{g_i} \cap B_1^{g_i} = (A_1 \cap B_1)^{g_i} = \sigma_1(M_\omega)^{g_i} = \sigma_i(M_\omega).$$

Therefore

$$\bigcap_{v \in V} K_v = \prod_{i=1}^k \sigma_i(M_\omega) = M_\omega.$$

Hence (1) holds. The choice of A_1 and B_1 is such that $T_1 = A_1 B_1$, and the definition of $K_v = \prod_i K_{v,i}$ implies that, for each i and v,

$$K_{v,i}\left(\bigcap_{v' \neq v} K_{v',i}\right) = T_i.$$

As $K_{v,i} \leqslant K_v$ for all i and v, it follows that $K_v(\bigcap_{v' \neq v} K_{v'}) = M$ for all v. Thus (2) holds and $\mathcal{K}$ is a Cartesian system for M with respect to ω. Now we prove that the set $\mathcal{K}$ is invariant under conjugation by G_ω. Let $v \in V$, $i_1, i_2 \in \{1, \ldots, k\}$ and $g \in G_\omega$ such that $T_{i_1}^g = T_{i_2}$. We claim that $K_{v,i_1}^g = K_{v^g,i_2}$. Suppose first that $v = v_1^{g_{i_1}}$. As g induces an automorphism of Γ, we obtain that $v_1^{g_{i_1} g} \in \varepsilon(T_{i_1}^g) = \varepsilon(T_{i_2}) = \{v_1^{g_{i_2}}, v_2^{g_{i_2}}\}$. If $v^g = v_1^{g_{i_1} g} = v_1^{g_{i_2}}$, then $g_{i_1} g g_{i_2}^{-1}$ stabilises (v_1, v_2), hence, by (10), normalises A_1 and B_1, so that $K_{v,i_1}^g = A_1^{g_{i_1} g} = A_1^{g_{i_2}}$ and $K_{v^g,i_2} = A_1^{g_{i_2}}$. Thus $K_{v,i_1}^g = K_{v^g,i_2}$. Similar arguments show that $K_{v,i_1}^g = K_{v^g,i_2}$ holds in all other cases. Therefore

$$K_v^g = \left(\prod_{i=1}^{k} K_{v,i}\right)^g = \prod_{i=1}^{k} K_{v^g,i} = K_{v^g}.$$

Hence $\mathcal{K}$ is G_ω-invariant. We also note that the G_ω-actions on V and on $\mathcal{K}$ are equivalent. Thus $\mathcal{K}$ is a G_ω-transitive Cartesian system of subgroups in M with respect to ω, and it follows from the definition of the K_v that $\mathcal{E}(\mathcal{K}) \in \mathsf{CD}_{2\sim}(G)$, $\Gamma \cong \Gamma(G, \mathcal{E}(\mathcal{K}))$ and $\mathcal{F}_1(\mathcal{E}(\mathcal{K}), M, \omega) = \{A_1, B_1\}$.

One aim of this section is to describe those innately transitive permutation groups G for which $\mathsf{CD}_{2\sim}(G)$ is non-empty. Our results show that, if $\mathsf{CD}_{2\sim}(G) \neq \emptyset$, then the following all hold: the isomorphism type of such groups is restricted (see Prop2 $\sim$[d]), and a point stabiliser in the plinth also satisfies some interesting properties, expressed in Prop2$\sim$[b]. Moreover, such groups G act on a generalised graph (see Prop2$\sim$[c]) whose edge set is intrinsic to the abstract group theoretic structure of G. This suggests that the conjugation action of G on the simple direct factors of the plinth may, in certain cases, predetermine the existence of Cartesian decompositions in $\mathsf{CD}_{2\sim}(G)$. This problem would be very interesting to address in more detail, but it would distract us from the main focus of the present work. We only illustrate this phenomenon with the following example

Example 5.3: Suppose that G is a quasiprimitive permutation group on Ω with a unique minimal normal subgroup $M = T_1 \times \cdots \times T_k$, where $k \geqslant 4$ and $T_1, \ldots, T_k$ are finite simple groups, isomorphic to one of the groups T in Table 2. Assume further that the conjugation action of G induces a 2-transitive permutation group on the set $\mathcal{T} = \{T_1, \ldots, T_k\}$. Let $\omega \in \Omega$. Then $G = MG_\omega$, and so the G_ω-action on $\mathcal{T}$ is also 2-transitive. Let $\Gamma = (V, \mathcal{T}, \varepsilon)$ be a generalised graph such that G_ω induces a vertex-transitive group of automorphisms on Γ where the G_ω-action on $\mathcal{T}$ is by conjugation. Then Lemma 3.3 implies that Γ is isomorphic to the union of k copies of the complete graph K_2. This shows that if $\mathcal{E} \in \mathsf{CD}_{2\sim}(G)$, then $|\mathcal{E}| = 2k$. Further, if K is a subgroup in the Cartesian system $\mathcal{K}_\omega(\mathcal{E})$, then K corresponds to a vertex of Γ that is adjacent to a unique edge of Γ. Thus there is a unique $i \in \{1,\dots,k\}$ such that $\sigma_i(K) \neq T_i$ and, since $\mathcal{K}_\omega(\mathcal{E})$ involves no strips, there is a unique i such that $T_i \not\leq K$. This shows that the corresponding embedding of G into the full stabiliser in $\mathsf{Sym}\,\Omega$ of $\mathcal{E}$ is as in [Theorem 1.1(b), BPS06].

6. Cartesian decompositions in $\mathsf{CD}_{2\not\sim}(G)$

In this section we assume that G is a finite innately transitive group acting on Ω with a non-abelian plinth $M = T_1 \times \cdots \times T_k$ where each of the T_i is isomorphic to a finite simple group T. Set $\mathcal{T} = \{T_1,\dots,T_k\}$, and fix an $\omega \in \Omega$.

Suppose that $\mathcal{E} \in \mathsf{CD}_{2\not\sim}(G)$ and let $\mathcal{K}_\omega(\mathcal{E}) = \{K_1,\dots,K_\ell\}$ be the corresponding Cartesian system of subgroups. For $i = 1,\dots,\ell$, set $\overline{K}_i = \sigma_1(K_i)\times\cdots\times\sigma_k(K_i)$, and let $\overline{\mathcal{K}}_\omega(\mathcal{E}) = \{\overline{K}_1,\dots,\overline{K}_\ell\}$. Let $\overline{M}_\omega = \overline{K}_1\cap\cdots\cap\overline{K}_\ell$, and note that $\overline{M}_\omega$ is the direct product of its projections, that is to say, $\overline{M}_\omega = \sigma_1(\overline{M}_\omega)\times\cdots\times\sigma_k(\overline{M}_\omega)$. Let $\overline{\Omega}$ denote the M-invariant partition $\mathbb{P}_M\left(\overline{M}_\omega\right)$ of Ω. It is routine to check that the conjugation action of G_ω permutes the subgroups $\overline{K}_1,\dots,\overline{K}_\ell$, and so their intersection $\overline{M}_\omega$ is invariant under G_ω. Thus Lemma 4.1 shows that $\overline{\Omega}$ is G-invariant. Let $\overline{\omega}$ be the block in $\overline{\Omega}$ that contains ω. Then $\overline{M}_\omega = M_{\overline{\omega}}$, and Lemma 4.1 also implies that $G_{\overline{\omega}} = \overline{M}_\omega G_\omega$. Let $\overline{G}$ denote the group induced by G on $\overline{\Omega}$, so that $\overline{G_{\overline{\omega}}}$ is the image in $\overline{G}$ of $G_{\overline{\omega}}$.

Define a generalised di-graph $\Gamma(G,\mathcal{E}) = (\mathcal{K}_\omega(\mathcal{E}),\mathcal{T},\beta,\varepsilon)$ for the given Cartesian decomposition $\mathcal{E}$ as follows. Let A_1 and B_1 be the subgroups of T_1 such that $\mathcal{F}_1(\mathcal{E},M,\omega) = \{A_1,B_1\}$. Then for each i there are unique indices j_1 and j_2 such that $\sigma_i(K_{j_1})$ is G_ω-conjugate to A_1 and $\sigma_i(K_{j_2})$ is G_ω-conjugate to B_1. Set $\beta(T_i) = K_{j_1}$ and $\varepsilon(T_i) = K_{j_2}$, and let A_i and B_i denote $\sigma_i(K_{j_1})$ and $\sigma_i(K_{j_2})$, respectively. Thus the subgroups $A_1,\dots,A_k$ are pairwise G_ω-conjugate, and so are the subgroups $B_1,\dots,B_k$. On the other hand, if i, $j \in \{1,\dots,k\}$, then A_i is not G_ω-conjugate to B_j.

The main result of this section is the following

THEOREM 6.1: *Let the groups G and M be as in the first paragraph of this section. If $\mathcal{E} \in \mathsf{CD}_{2\not\sim}(G)$, then the properties* Prop2≁[a]–Prop2≁[c] *below, all hold.*

Prop2≁[a] (Quotient Action Property)**.** The group M is faithful on $\overline{\Omega}$, and so, if K is a subgroup of M, then we identify K with its image under the action on $\overline{\Omega}$. The set $\overline{\mathcal{K}}_\omega(\mathcal{E})$ is a $\overline{G_{\overline{\omega}}}$-invariant Cartesian system of subgroups for M with respect to $\overline{\omega}$. Moreover, $\mathcal{E}(\overline{\mathcal{K}}_\omega(\mathcal{E})) \in \mathsf{CD}_{2\not\sim}(\overline{G})$.

Prop2$\not\sim$[b] (Factorisation Property)**.** If $i \in \{1, \ldots, k\}$ then

(i) A_i, B_i are proper subgroups of T_i;

(ii) A_i and B_i are not conjugate under G_ω;

(iii) $A_i B_i = T_i$ and $A_i \cap B_i = \sigma_i(\overline{M}_\omega)$;

(iv) $\mathbb{N}_{G_\omega}(T_i) = \mathbb{N}_{G_\omega}(A_i) = \mathbb{N}_{G_\omega}(B_i)$.

Prop2$\not\sim$[c] (Combinatorial Property)**.** The group G_ω induces a group of automorphisms of the generalised di-graph $\Gamma(G, \mathcal{E})$, which is transitive on both the vertex-set $\mathcal{K}_\omega(\mathcal{E})$ and the arc-set $\mathcal{T}$, where the G_ω-actions are by conjugation.

The observant reader may notice that our conclusions in this section are considerably weaker than those in Section 5, as there is no counterpart of Prop2$\sim$[d]. The reason for this is simple: in the previous section the finite simple group T admitted a factorisation with two proper, isomorphic subgroups, and so the isomorphism type of T, and hence that of G, could be restricted. No such factorisation is guaranteed to exist here. On the other hand, for some i, the subgroups A_i and B_i may be isomorphic even though they are not G_ω-conjugate. It is easy to see, and is left to the reader, that claims similar to those in Prop2$\sim$[d] are valid in this case. We formulate the following related problem.

Problem: *Let G, M and ω be as in the first paragraph of this section and let $\mathcal{E} \in \mathsf{CD}_{2\not\sim}(G)$ such that $\mathcal{F}_1(\mathcal{E}, M, \omega)$ contains two isomorphic subgroups. Is it always true that there is an innately transitive subgroup H of $\mathsf{Sym}\,\Omega$, having the same plinth M as G, such that $\mathcal{E} \in \mathsf{CD}_{2\sim}(H)$?*

Next we prove Theorem 6.1.

6.1. Proof of Theorem 6.1. **Prop2$\not\sim$[a]** As $\mathcal{E} \in \mathsf{CD}_{2\not\sim}(G)$ we have that, for all i, there are two indices j such that $\sigma_i(K_j) < T_i$. Thus each of the $\overline{K}_i$ is a proper subgroup of M. This also shows that $\sigma_i(\overline{M}_\omega)$ is a proper subgroup of T_i for all $i \in \{1, \ldots, k\}$. Thus no T_i is a subgroup of $\overline{M}_\omega$, and so M must be faithful on $\overline{\Omega}$. Set $\mathcal{K} = \mathcal{K}_\omega(\mathcal{E})$ and $\overline{\mathcal{K}} = \overline{\mathcal{K}}_\omega(\mathcal{E})$.

Next we prove that $\overline{\mathcal{K}}$ is a $\overline{G}_{\overline{\omega}}$-invariant Cartesian system for M with respect to $\overline{\omega}$. Equation (1) holds because of the definition of $\overline{M}_\omega = M_{\overline{\omega}}$. Since (2) holds for $\mathcal{K}$ and $K_i \leqslant \overline{K}_i$ for all i, we obtain that (2) holds for $\overline{\mathcal{K}}$ as well. We claim that $\overline{\mathcal{K}}$ is invariant under conjugation by G_ω. Let $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, \ell\}$, and $g \in G_\omega$. We denote by i^g the integer in $\{1, \ldots, k\}$ that satisfies $T_i^g = T_{i^g}$. Then $\sigma_i(K_j)^g = \sigma_{i^g}(K_j^g)$. Thus

$$(\overline{K}_j)^g = \left(\prod_{i=1}^{k} \sigma_i(K_j)\right)^g = \prod_{i=1}^{k} \sigma_{i^g}(K_j^g) = \prod_{i=1}^{k} \sigma_i(K_j^g).$$

Since $K_j^g \in \mathcal{K}$, it follows that $(\overline{K}_j)^g \in \overline{\mathcal{K}}$. Lemma 4.1 shows that $G_{\overline{\omega}} = \overline{M}_\omega G_\omega$. Then, since $\overline{M}_\omega = \overline{K}_1 \cap \cdots \cap \overline{K}_\ell$ preserves $\overline{\mathcal{K}}$, and $\overline{\mathcal{K}}$ is G_ω-invariant, we obtain that $\overline{\mathcal{K}}$ is also $G_{\overline{\omega}}$-invariant. Thus $\overline{\mathcal{K}}$ is a $\overline{G}_{\overline{\omega}}$-invariant Cartesian system of subgroups for M with respect to $\overline{\omega}$.

It follows from the definition of $\overline{\mathcal{K}}$ that $\mathcal{F}_i(\mathcal{E}, M, \omega) = \mathcal{F}_i(\mathcal{E}(\overline{\mathcal{K}}), M, \overline{\omega}) = \{A_i, B_i\}$ for all $i \in \{1, \ldots, k\}$. Let $\overline{g} \in \overline{G}_{\overline{\omega}}$ such that $A_1^{\overline{g}} = B_1$ and let g denote its preimage in $G_{\overline{\omega}}$. Then $g = mg_1$ for some $m \in \overline{M}_\omega$ and $g_1 \in G_\omega$. As $\overline{M}_\omega$ is the intersection of the $\overline{K}_j$, we obtain that

$$\begin{aligned}\sigma_1(m) \in \sigma_1(\overline{K}_1 \cap \cdots \cap \overline{K}_\ell) &\leqslant \sigma_1(\overline{K}_1) \cap \cdots \cap \sigma_1(\overline{K}_\ell) \\ &= \sigma_1(K_1) \cap \cdots \cap \sigma_1(K_\ell) = A_1 \cap B_1.\end{aligned}$$

Therefore $A_1^m = A_1$, and so $A_1^{g_1} = B_1$. As $g_1 \in G_\omega$ and $\mathcal{E} \in \mathsf{CD}_{2\not\sim}(G)$, this is a contradiction, and so A_1 and B_1 are not $\overline{G}_{\overline{\omega}}$-conjugate. Thus $\mathcal{E}(\overline{\mathcal{K}}) \in \mathsf{CD}_{2\not\sim}(\overline{G})$.

Prop2$\not\sim$[b] Let $i \in \{1, \ldots, k\}$ and let $j_1,\ j_2 \in \{1, \ldots, \ell\}$ be such that $A_i = \sigma_i(K_{j_1})$ and $B_i = \sigma_i(K_{j_2})$. By the definition of $\mathsf{CD}_{2\not\sim}(G)$, it follows that Prop2$\not\sim$[b](i)–(ii) hold for A_i and B_i. Since $K_{j_1}K_{j_2} = M$ we have that $\sigma_i(K_{j_1})\sigma_i(K_{j_2}) = \sigma_i(M)$ and so $A_iB_i = T_i$. Also,

$$\begin{aligned}A_i \cap B_i &= \sigma_i(K_{j_1}) \cap \sigma_i(K_{j_2}) = \sigma_i(\overline{K}_{j_1}) \cap \sigma_i(\overline{K}_{j_2}) \\ &= \sigma_i(\overline{K}_1) \cap \cdots \cap \sigma_i(\overline{K}_\ell) = \sigma_i(\overline{K}_1 \cap \cdots \cap \overline{K}_\ell) = \sigma_i(\overline{M}_\omega).\end{aligned}$$

Hence Prop2$\not\sim$[b](iii) is valid. If $g \in \mathbb{N}_{G_\omega}(A_i)$ then $T_i^g \cap T_i \neq 1$ and so $T_i^g = T_i$. Thus $g \in \mathbb{N}_{G_\omega}(T_i)$, and so $\mathbb{N}_{G_\omega}(A_i) \leqslant \mathbb{N}_{G_\omega}(T_i)$. Similarly $\mathbb{N}_{G_\omega}(B_i) \leqslant \mathbb{N}_{G_\omega}(T_i)$. Suppose now that $g \in \mathbb{N}_{G_\omega}(T_i)$. Then $\sigma_i(K)^g = \sigma_i(K^g)$ for all $K \in \mathcal{K}$. Thus the uniqueness of j_1 and j_2 yields that g fixes K_{j_1} and K_{j_2}. Therefore g fixes $\sigma_i(K_{j_1})$ and $\sigma_i(K_{j_2})$, and so $g \in \mathbb{N}_{G_\omega}(A_i) \cap \mathbb{N}_{G_\omega}(B_i)$. Thus all properties in Prop2$\not\sim$[b] hold.

Prop2$\not\sim$[c] Let Γ denote $\Gamma(G, \mathcal{E})$. It follows from the definition of Γ that the conjugation action of G_ω is transitive on the vertex-set $\mathcal{K}$ and on the arc-set $\mathcal{T}$ of Γ. We claim that G_ω preserves adjacency in $\Gamma(G, \mathcal{E})$. Let $i_1 \in \{1, \ldots, k\}$ with $\beta(T_{i_1}) = K_j$, so that $\sigma_{i_1}(K_j)$ is G_ω-conjugate to A_1, and let $g \in G_\omega$. Let $i_2 \in \{1, \ldots, k\}$ such that $T_{i_1}^g = T_{i_2}$. Then $\sigma_{i_1}(K_j)^g = \sigma_{i_2}(K_j^g)$. As $\sigma_{i_1}(K_j)$ is G_ω-conjugate to A_1, so is $\sigma_{i_2}(K_j^g)$. Thus $\beta(T_{i_1}^g) = \beta(T_{i_2}) = K_j^g = \beta(T_{i_1})^g$, as required. Thus β is preserved by the G_ω-action; similar argument shows that ε is also preserved by the G_ω-action. Hence all claims of the theorem hold. ∎

6.2. A converse of Theorem 6.1. Theorem 6.1 can be reversed in the following sense. Let G be a finite innately transitive group on Ω with a non-abelian plinth M, and let $T_1, \ldots, T_k$ be the simple direct factors of M. Assume that a point stabiliser M_ω can be decomposed as $M_\omega = \sigma_1(M_\omega) \times \cdots \times \sigma_k(M_\omega)$. Set $\mathcal{T} = \{T_1, \ldots, T_k\}$. Suppose that $A_1,\ B_1$ are subgroups of T_1 and $\Gamma = (V, \mathcal{T}, \beta, \varepsilon)$ is a generalised di-graph, such that properties Prop2$\not\sim$[b] and Prop2$\not\sim$[c] hold. This amounts to saying that

(i) $A_1,\ B_1$ are proper subgroups of T_1;
(ii) A_1 and B_1 are not conjugate under G_ω;
(iii) $A_1 B_1 = T_1$, $A_1 \cap B_1 = \sigma_1(M_\omega)$;
(iv) $\mathbb{N}_{G_\omega}(T_1) = \mathbb{N}_{G_\omega}(A_1) = \mathbb{N}_{G_\omega}(B_1)$;

and also that G_ω induces a vertex and arc-transitive group of automorphisms of Γ, where the G_ω-actions are by conjugation.

For $i = 1, \ldots, k$, choose $g_i \in G_\omega$ such that $T_1^{g_i} = T_i$. For each element $v \in V$ set $K_v = \prod_{i=1}^k K_{v,i}$ where

$$K_{v,i} = \begin{cases} A_1^{g_i} & \text{if } \beta(T_i) = v; \\ B_1^{g_i} & \text{if } \varepsilon(T_i) = v; \\ T_i & \text{otherwise.} \end{cases}$$

First we prove that the $K_{v,i}$ are well-defined, that is, their definitions are independent of the choice of the g_i. Suppose that $g_i,\ g_i' \in G_\omega$ are such that $T_1^{g_i} = T_1^{g_i'} = T_i$. Then $g_i g_i'^{-1} \in \mathbb{N}_{G_\omega}(T_1)$ and so, by property (iv) above, $g_i g_i'^{-1} \in \mathbb{N}_{G_\omega}(A_1) \cap \mathbb{N}_{G_\omega}(B_1)$. Hence $A_1^{g_i} = A_1^{g_i'}$ and $B_1^{g_i} = B_1^{g_i'}$. Thus the $K_{v,i}$ are well-defined.

Let $\mathcal{K} = \{K_v |\ v \in V\}$. We claim that $\mathcal{K}$ is a G_ω-invariant Cartesian system for M with respect to ω. First note that the K_v are direct products of their projections and, for all i,

$$\bigcap_{v \in V} K_{v,i} = A_1^{g_i} \cap B_1^{g_i} = (A_1 \cap B_1)^{g_i} = \sigma_1(M_\omega)^{g_i} = \sigma_i(M_\omega).$$

Therefore

$$\bigcap_{v \in V} K_v = \prod_{i=1}^k \sigma_i(M_\omega) = M_\omega.$$

Hence (1) holds. The choice of A_1 and B_1 and the definition of the $K_{v,i}$ imply that for each i and v,

$$K_{v,i}\Bigg(\bigcap_{v' \neq v} K_{v',i}\Bigg) = T_i.$$

As $K_{v,i} \leqslant K_v$ for all i and v, it follows that $K_v(\bigcap_{v'\neq v} K_{v'}) = M$ for all v. Thus (2) holds and $\mathcal{K}$ is a Cartesian system for M with respect to ω. We prove now that the set $\mathcal{K}$ is invariant under conjugation by G_ω. Let $v \in V$, $i_1, i_2 \in \{1,\ldots,k\}$ and $g \in G_\omega$ such that $T_{i_1}^g = T_{i_2}$. We claim that $K_{v,i_1}^g = K_{v^g,i_2}$. Suppose first that $\beta(T_{i_1}) = v$. Then, as g induces an automorphism of Γ, we obtain that $\beta(T_{i_1}^g) = v^g$, that is, $\beta(T_{i_2}) = v^g$. Thus in this case we have $K_{v^g,i_2} = A_1^{g_{i_2}}$, $K_{v,i_1} = A_1^{g_{i_1}}$, and hence $K_{v,i_1}^g = A_1^{g_{i_1}g}$. As $T_1^{g_{i_1}g} = T_{i_1}^g = T_{i_2}$, we obtain, as above, that $A_1^{g_{i_1}g} = A_1^{g_{i_2}}$. Hence $K_{v,i_1}^g = K_{v^g,i_2}$. Similar argument shows that this equality also holds when $\varepsilon(T_{i_1}) = v$, and when T_{i_1} is not adjacent to v. Therefore

$$K_v^g = \left(\prod_{i=1}^k K_{v,i}\right)^g = \prod_{i=1}^k K_{v^g,i} = K_{v^g}.$$

Hence $\mathcal{K}$ is G_ω-invariant. The last equation also shows that the G_ω-actions on V and on $\mathcal{K}$ are equivalent. Thus $\mathcal{K}$ is a G_ω-transitive Cartesian system of subgroups for M with respect to ω, and it follows from the definition of the K_v that $\mathcal{E}(\mathcal{K}) \in \mathsf{CD}_{2\not\sim}(G)$, $\Gamma = \Gamma(G,\mathcal{E})$ and $\mathcal{F}_i(\mathcal{E}(\mathcal{K}), M, \omega) = \{A_i, B_i\}$ for each i.

As in the previous section, it is possible to investigate further the conditions that ensure the existence of a Cartesian decomposition in $\mathsf{CD}_{2\not\sim}(G)$ for some innately transitive group G. If this set of decompositions is non-empty then the set of simple direct factors of the plinth must play the rôle of the arc-set of a generalised di-graph. Thus we expect that the nature of the conjugation action on the simple direct factors can strongly restrict the possible generalised di-graphs satisfying $\mathsf{Prop2}{\not\sim}[\mathsf{c}]$, and hence the possible elements of $\mathsf{CD}_{2\not\sim}(G)$. Though details of this phenomenon are not addressed in this paper, we present a simple example for illustration.

Example 6.2: We claim that no generalised di-graph exists having four arcs and admitting an automorphism group that acts vertex and arc-transitively inducing an A_4 or S_4 on the arcs. For, if Γ is such a generalised di-graph then every vertex has a constant number of outgoing arcs. Hence the number of vertices must be a divisor of 4 (and is not 1 by the definition of a generalised di-graph). It is left to the reader to check that no such graph exists on 2 or 4 vertices. Therefore if G is a finite innately transitive group with a non-abelian plinth $M = T_1 \times \cdots \times T_4$ where the permutation action of G on the T_i is permutationally isomorphic to A_4 or S_4 then $\mathsf{CD}_{2\not\sim}(G) = \emptyset$.

7. Cartesian decompositions in $\mathsf{CD}_{1S}(G)$

In this section the following notation is used. Let G be a finite innately transitive group on Ω with a non-abelian plinth M, and let $T_1,\ldots,T_k$ be the simple normal subgroups of M, each isomorphic to the simple group T. Let $\omega\in\Omega$, and set $\mathcal{T}=\{T_1,\ldots,T_k\}$. Let $\overline{M}_\omega=\mathbb{N}_M(M_\omega)$ and let $\overline{\Omega}$ denote the M-invariant partition $\mathbb{P}_M\left(\overline{M}_\omega\right)$ of Ω. Then $\overline{M}_\omega$ is normalised by G_ω. Thus Lemma 4.1 shows that $\overline{\Omega}$ is G-invariant. Let $\overline{\omega}$ be the block in $\overline{\Omega}$ that contains ω. Then Lemma 4.1 also implies that $G_{\overline{\omega}}=\overline{M}_\omega G_\omega$, that $\overline{\omega}$ is the $\overline{M}_\omega$-orbit containing ω, and that $\overline{M}_\omega=M_{\overline{\omega}}$. Let $\overline{G}$ denote the permutation group on $\overline{\Omega}$ induced by G, so $\overline{G}_{\overline{\omega}}$ is the subgroup of $\overline{G}$ induced by $G_{\overline{\omega}}$.

Suppose that $\mathcal{E}\in\mathsf{CD}_{1S}(G)$ and let $\mathcal{K}_\omega(\mathcal{E})=\{K_1,\ldots,K_\ell\}$ be the corresponding Cartesian system. For $K\in\mathcal{K}_\omega(\mathcal{E})$, let $\mathcal{X}_K$ denote the set of non-trivial, full strips involved in K, and set $\mathcal{X}=\mathcal{X}_{K_1}\cup\cdots\cup\mathcal{X}_{K_\ell}$. By Theorem 2.2, $\mathcal{X}$ contains $k/2$ pairwise disjoint, full strips, each of length 2. Let

$$\overline{K}_i=\prod_{X\in\mathcal{X}_{K_i}}X\times\prod_{T_m\notin\bigcup_{X\in\mathcal{X}_{K_i}}\operatorname{Supp}X}\sigma_m(K_i).$$

Set $\overline{\mathcal{K}}_\omega(\mathcal{E})=\{\overline{K}_1,\ldots,\overline{K}_\ell\}$. If X is a strip in M then we define $\min X=\min\{i|\ T_i\in\operatorname{Supp}X\}$ and $\max X=\max\{i|\ T_i\in\operatorname{Supp}X\}$. Suppose that $\mathcal{X}=\{X_1,\ldots,X_{k/2}\}$, and, for $i=1,\ldots,k/2$, let A_i and B_i be defined as follows. There are unique indices j_1 and j_2 such that $\sigma_{\min X_i}(K_{j_1})\neq T_{\min X_i}$ and $\sigma_{\max X_i}(K_{j_2})\neq T_{\max X_i}$; set $A_i=\sigma_{\min X_i}(K_{j_1})$ and $B_i=\sigma_{\max X_i}(K_{j_2})$. Let $\Gamma(G,\mathcal{E})$ denote the graph $(\mathcal{K}_\omega(\mathcal{E})\cup\mathcal{X},E_1\cup E_2)$ where, for $K\in\mathcal{K}_\omega(\mathcal{E})$ and $X\in\mathcal{X}$, $\{K,X\}\in E_1$ if either $\sigma_{\min X}(K)<T_{\min X}$ or $\sigma_{\max X}(K)<T_{\max X}$, and $\{K,X\}\in E_2$ if X is involved in K.

Theorem 7.1: *Let G and M be as in the first paragraph of this section. If $\mathcal{E}\in\mathsf{CD}_{1S}(G)$, then the properties* Prop1S[a]–Prop1S[d] *below all hold.*

Prop1S[a] (Quotient Action Property**).** The group M is faithful on $\overline{\Omega}$, and so, if K is a subgroup of M, then we identify K with its image under the action on $\overline{\Omega}$. The set $\overline{\mathcal{K}}_\omega(\mathcal{E})$ is a $\overline{G}_{\overline{\omega}}$-invariant Cartesian system of subgroups for M with respect to $\overline{\omega}$. Moreover, $\mathcal{E}(\overline{\mathcal{K}}_\omega(\mathcal{E}))\in\mathsf{CD}_{1S}(\overline{G})$.

Prop1S[b] (Factorisation Property**).** If $i\in\{1,\ldots,k/2\}$ then

(i) X_i is a full strip of length 2;
(ii) A_i is a proper subgroup of $T_{\min X_i}$ and B_i is a proper subgroup of $T_{\max X_i}$;
(iii) A_i and B_i are conjugate under G_ω;
(iv) $X_i(A_i\times B_i)=T_{\min X_i}\times T_{\max X_i}$, $X_i\cap(A_i\times B_i)=\sigma_{\operatorname{Supp}X}(\mathbb{N}_M(M_\omega))$;

(v) $\mathbb{N}_{G_\omega}(T_{\min X_i} \times T_{\max X_i}) = \mathbb{N}_{G_\omega}(X_i) = \mathbb{N}_{G_\omega}(A_i \times B_i)$.

Prop1S[c] (Combinatorial Property**).** The graph $\Gamma(G, \mathcal{E})$ is bipartite, with bipartition formed by the sets $\mathcal{K}_\omega(\mathcal{E})$ and $\mathcal{X}$. The group G_ω induces a group of automorphisms of the graph $\Gamma(G, \mathcal{E})$, such that $\mathcal{K}_\omega(\mathcal{E})$, $\mathcal{X}$, E_1 and E_2 are G-orbits. Moreover, each element of $\mathcal{X}$ is adjacent to one edge or two edges from E_1, and one edge of E_2. Further, if the elements of $\mathcal{X}$ are adjacent to two elements of E_1 then the following condition must also hold: if, for some $K \in \mathcal{K}_\omega(\mathcal{E})$ and $X \in \mathcal{X}$, $\{K, X\} \in E_1$ then $(G_\omega)_{\{K,X\}} = \mathbb{N}_{G_\omega}(T_{\min X}) \cap \mathbb{N}_{G_\omega}(T_{\max X})$.

Prop1S[d] (Isomorphism Property**).** The group T, the subgroup A_i (which is G_ω-conjugate to B_i), and $\sigma_{\mathrm{Supp}\, X_i}(M_{\overline{\omega}})$ are as in Table 3. The group $\overline{G}$ is permutationally isomorphic to a subgroup of $\mathsf{Aut}\, M$ acting on $\overline{\Omega}$. In particular, M is the unique minimal normal subgroup of $\overline{G}$, and $\overline{G}$ is quasiprimitive. Moreover, if T is as in rows 1–3 of Table 3 then $\overline{M}_\omega = M_\omega$, and $G \cong \overline{G}$ as permutation groups. Otherwise each block in $\overline{\Omega}$ has size dividing $2^{k/2}$, the kernel N of the action of G on $\overline{\Omega}$ is an elementary abelian 2-group of rank at most $k/2$ and $\overline{G} \cong G/N$.

7.1. Some computation. The following three results are needed for the proof of Theorem 7.1. The results are stated and proved in the context introduced in the first two paragraphs of this section.

Proposition 7.2: *If $\mathcal{E} \in \mathsf{CD}_{1\mathrm{S}}(G)$, then the group T is as in one of the rows of Table 3, and each $\mathcal{F}_i(\mathcal{E}, M, \omega)$ contains a subgroup isomorphic to the group A_i in the corresponding row of Table 3. Further, for all $K \in \mathcal{K}_\omega(\mathcal{E})$,*

$$\prod_{X \in \mathcal{X}_K} X \times \prod_{T_m \notin \bigcup_{X \in \mathcal{X}_K} \mathrm{Supp}\, X} \sigma_m(K)' \leqslant K. \tag{11}$$

If T is as in rows 1–3 then

$$K = \prod_{X \in \mathcal{X}_K} X \times \prod_{T_m \notin \bigcup_{X \in \mathcal{X}_K} \mathrm{Supp}\, X} \sigma_m(K) \quad \text{for all } K \in \mathcal{K}_\omega(\mathcal{E}). \tag{12}$$

	T	A_i	$\sigma_{\mathrm{Supp}\, X_i}(M_{\overline{\omega}})$
1	A_6	A_5	D_{10}
2	M_{12}	M_{11}	$\mathsf{PSL}_2(11)$
3	$\mathsf{P}\Omega_8^+(q)$	$\Omega_7(q)$	$\mathsf{G}_2(q)$
4	$\mathsf{Sp}_4(q)$, $q \geqslant 4$ even	$\mathsf{Sp}_2(q^2) \cdot 2$	$\mathsf{D}_{q^2+1} \cdot 2$

Table 3. Factorisations of finite simple groups with two isomorphic subgroups

Proof: Let $\mathcal{K}_\omega(\mathcal{E}) = \{K_1, \ldots, K_\ell\}$, and for $i = 1, \ldots, \ell$ let $\widehat{K}_i$ denote $\bigcap_{j \neq i} K_j$. By Theorem 2.2, each non-trivial, full strip involved in a K_i has length 2. Suppose without loss of generality that X is a non-trivial full strip involved in K_1 covering T_1 and T_2. Thus by (2), $T_1 \times T_2 = \sigma_{\{1,2\}}(K_1)\sigma_{\{1,2\}}(\widehat{K}_1)$. Suppose that $X = \{(t, \alpha(t)) | \ t \in T_1\}$ for some isomorphism $\alpha : T_1 \to T_2$. Then it follows from [Lemma 2.1, PS02] that $T_1 = \sigma_1(\widehat{K}_1)\alpha^{-1}(\sigma_2(\widehat{K}_1))$. By the definition of $\mathsf{CD}_{1S}(G)$, $\sigma_1(K_{j_1}) \neq T_1$ and $\sigma_2(K_{j_2}) \neq T_2$ for some $j_1, \ j_2 \in \{2, \ldots, \ell\}$. Thus $\sigma_1(\widehat{K}_1)$ and $\sigma_2(\widehat{K}_1)$ are proper subgroups of T_1 and T_2, respectively. Moreover, if $g \in G_\omega$ such that $T_1^g = T_2$ then $T_2 \in \operatorname{Supp} X \cap \operatorname{Supp} X^g$, and so Theorem 2.2 implies that $X^g = X$. Hence, again by Theorem 2.2, $g \in \mathbb{N}_{G_\omega}(K_1)$, and also $g \in \mathbb{N}_{G_\omega}\left(\widehat{K}_1\right)$. Thus $\sigma_1(\widehat{K}_1)^g = \sigma_2(\widehat{K}_1)$ is a proper subgroup of T_2. Hence $T_1 = \sigma_1(\widehat{K}_1)\alpha^{-1}(\sigma_2(\widehat{K}_1))$ is a factorisation with proper, isomorphic subgroups. Therefore [Lemma 5.2, BPS04] implies that $T_1 \cong T$ is as in Table 3 and the isomorphism types of $\sigma_1(\widehat{K}_1)$ and $\sigma_2(\widehat{K}_1)$ are as in the A_i-column of the corresponding row of Table 3.

Suppose that $\sigma_i(K_j) \neq T_i$ for some i and j. Then there is a non-trivial full strip $X \in \mathcal{X}$ covering T_i; assume that $X \in \mathcal{X}_{K_m}$ for some $m \in \{1, \ldots, \ell\} \setminus \{j\}$ and that $\operatorname{Supp} X = \{T_i, T_{i'}\}$. Then the argument of the previous paragraph shows that $\sigma_i(\widehat{K}_m)$ is as in the A_i-column of Table 3. In particular, $\sigma_i(\widehat{K}_m)$ is a maximal subgroup of T_i. Since $\widehat{K}_m \leqslant K_j$, we obtain that $\sigma_i(\widehat{K}_m) \leqslant \sigma_i(K_j) < T_i$, and so $\sigma_i(\widehat{K}_m) = \sigma_i(K_j)$. Therefore $\sigma_i(K_j)$ is also as in the A_i-column of the corresponding row of the table.

We have proved so far that for all i and j either $\sigma_i(K_j) = T_i$ or $\sigma_i(K_j) \cong A$ where A is as in the A_i-column of Table 3. In particular A is a maximal subgroup of T, A is almost simple, and if T is as in rows 1–3 of Table 3 then A is simple. Suppose by contradiction that (11) fails to hold for some K_j. Set

$$\overline{S} = \left\{ m | \ T_m \in \bigcup_{X \in \mathcal{X}_{K_j}} \operatorname{Supp} X \right\} \quad \text{and} \quad S = \{1, \ldots, k\} \setminus \overline{S},$$

and write $\sigma_S, \sigma_{\overline{S}}$ for the projection of M onto $\prod_{s \in S} T_s$ and $\prod_{s \in \overline{S}} T_s$ respectively. Then it follows from the definition of $\mathcal{X}_{K_j}$ that

$$K_j = \sigma_S(K_j) \times \sigma_{\overline{S}}(K_j).$$

As (11) fails for K_j we must have that

$$\prod_{m \in S} \sigma_m(K_j)' \not\leqslant \sigma_S(K_j).$$

Thus it follows from [Lemma 2.3, PS02] that there are distinct elements i_1, i_2 of S such that

$$\sigma_{i_1}(K_j)' \times \sigma_{i_2}(K_j)' \nleqslant \sigma_{\{i_1,i_2\}}(K_j). \tag{13}$$

If $\sigma_{i_1}(K_j) = T_{i_1}$ then, by [Lemma 4.3, BPS06], there is a full strip X involved in K_j covering T_{i_1}. By the definition of S, we must have that $X = T_{i_1}$, and so $\sigma_{i_1}(K_j) \leqslant K_j$. Hence $\sigma_{i_1}(K_j) \leqslant \sigma_{\{i_1,i_2\}}(K_j)$, and also $\sigma_{i_2}(K_j) \leqslant \sigma_{\{i_1,i_2\}}(K_j)$. Hence $\sigma_{i_1}(K_j) \times \sigma_{i_2}(K_j) = \sigma_{\{i_1,i_2\}}(K_j)$, contradicting (13). Thus $\sigma_{i_1}(K_j)$ is a proper subgroup of T_{i_1} and also $\sigma_{i_2}(K_j)$ is a proper subgroup of T_{i_2}.

By Theorem 2.2(d), G_ω is transitive on $\mathcal{X}$, and so there are (not necessarily distinct) strips X_1 and X_2 in $\mathcal{X}$ such that X_1 covers T_{i_1} and X_2 covers T_{i_2}. Suppose that $X_1 = X_2$. Then $\operatorname{Supp} X_1 = \{T_{i_1}, T_{i_2}\}$, and let $j_1 \in \{1,\ldots,\ell\} \setminus \{j\}$ be such that $X_1 \in \mathcal{X}_{K_{j_1}}$. Then, as verified above, $\sigma_{i_1}(\widehat{K}_{j_1})$ and $\sigma_{i_2}(\widehat{K}_{j_1})$ are maximal subgroups of T_{i_1} and T_{i_2}, respectively, and, in addition, $\sigma_{i_1}(\widehat{K}_{j_1}) \cong \sigma_{i_2}(\widehat{K}_{j_1})$. Thus the factorisation

$$X_1\sigma_{\{i_1,i_2\}}(\widehat{K}_{j_1}) = \sigma_{\{i_1,i_2\}}(K_{j_1})\sigma_{\{i_1,i_2\}}(\widehat{K}_{j_1}) = T_{i_1} \times T_{i_2}$$

is as in [Theorem 1.5, PS02]. Hence [Theorem 1.5, PS02] implies that

$$\sigma_{i_1}(\widehat{K}_{j_1})' \times \sigma_{i_2}(\widehat{K}_{j_1})' \leqslant \sigma_{\{i_1,i_2\}}(\widehat{K}_{j_1}).$$

Note that $j \neq j_1$, and so $\sigma_{\{i_1,i_2\}}(\widehat{K}_{j_1}) \leqslant \sigma_{\{i_1,i_2\}}(K_j)$. Moreover, $\sigma_{i_1}(\widehat{K}_{j_1})$ is a maximal subgroup of T_{i_1} and so is $\sigma_{i_1}(K_j)$. As $\sigma_{i_1}(\widehat{K}_{j_1}) \leqslant \sigma_{i_1}(K_j)$, we obtain that $\sigma_{i_1}(\widehat{K}_{j_1}) = \sigma_{i_1}(K_j)$, and, similarly, $\sigma_{i_2}(K_{j_1}) = \sigma_{i_2}(K_j)$. Therefore

$$\sigma_{i_1}(K_j)' \times \sigma_{i_2}(K_j)' \leqslant \sigma_{\{i_1,i_2\}}(K_j),$$

which is a contradiction. Hence $X_1 \neq X_2$.

Suppose that X_1 is involved in K_{j_1} and X_2 is involved in K_{j_2}, where j_1 and j_2 are not necessarily distinct elements of $\{1,\ldots,\ell\} \setminus \{j\}$. Let $I = \operatorname{Supp} X_1 \cup \operatorname{Supp} X_2$ and set $\widehat{K}_{j_1,j_2} = \bigcap_{m\neq j_1,\ j_2} K_m$. Then, by [Lemma 3.1, BPS04], $(K_{j_1} \cap K_{j_2})\widehat{K}_{j_1,j_2} = M$, and so

$$\sigma_I(M) = \sigma_I(K_{j_1} \cap K_{j_2})\sigma_I(\widehat{K}_{j_1,j_2}).$$

Suppose that $n \in \operatorname{Supp} X_1 \cup \operatorname{Supp} X_2$; in fact suppose without loss of generality that $n \in \operatorname{Supp} X_1$. Then the argument above shows that $\sigma_n(\widehat{K}_{j_1}) \cong A$ and also $\sigma_n(K_{j'}) \cong A$ where A is as in the A_i-column of Table 3 and $j' \in \{1,\ldots,\ell\}$ is such that $\sigma_n(K_{j'}) < T_n$. Since,

$$\sigma_n(\widehat{K}_{j_1}) \leqslant \sigma_n(\widehat{K}_{j_1,j_2}) \leqslant \sigma_n(K_{j'}),$$

we obtain that $\sigma_n(\widehat{K}_{j_1,j_2}) \cong A$, and this holds for all $n \in \operatorname{Supp} X_1 \cup \operatorname{Supp} X_2$. Clearly $\sigma_I(K_{j_1} \cap K_{j_2}) \leqslant X_1 \times X_2$, and so

$$\sigma_I(M) = (X_1 \times X_2)\sigma_I(\widehat{K}_{j_1,j_2}).$$

Then it follows from [Theorem 1.5, PS02] that

$$\sigma_{\min X_1}(\widehat{K}_{j_1,j_2})' \times \sigma_{\max X_1}(\widehat{K}_{j_1,j_2})' \times \sigma_{\min X_2}(\widehat{K}_{j_1,j_2})' \times \sigma_{\max X_2}(\widehat{K}_{j_1,j_2})' \leqslant \sigma_I(\widehat{K}_{j_1,j_2}).$$

As $i_1,\ i_2 \in I$, we obtain that

$$\sigma_{i_1}(\widehat{K}_{j_1,j_2})' \times \sigma_{i_2}(\widehat{K}_{j_1,j_2})' \leqslant \sigma_{\{i_1,i_2\}}(\widehat{K}_{j_1,j_2}) \leqslant \sigma_{\{i_1,i_2\}}(K_j).$$

Since $\sigma_{i_1}(\widehat{K}_{j_1,j_2})' = \sigma_{i_1}(K_j)'$ and $\sigma_{i_2}(\widehat{K}_{j_1,j_2})' = \sigma_{i_2}(K_j)'$, this is a contradiction. Hence (11) holds. If T is as in rows 1–3, then $\sigma_i(K_j)$ is simple and hence perfect. This proves (12). ∎

Next we need to compute normalisers of point stabilisers and Cartesian system elements.

LEMMA 7.3: *If $\mathcal{E} \in \mathsf{CD}_{1\mathrm{S}}(G)$, then for all $i \in \{1, \ldots, \ell\}$,*

$$\mathbb{N}_M\ (K_i) = \overline{K}_i \quad \textit{and} \quad K_i' = \prod_{X \in \mathcal{X}_{K_i}} X \times \prod_{T_j \notin \bigcup_{X \in \mathcal{X}_{K_i}} \operatorname{Supp} X} \sigma_j(K_i)'.$$

Proof: If T is as in rows 1–3 of Table 3 then the claim of the lemma follows from the fact that, by Lemma 3.5, each strip $X \in \mathcal{X}_{K_i}$ is a simple and self-normalising subgroup of $\sigma_{\operatorname{Supp} X}(M)$, and if T_j is not covered by any strip in $\mathcal{X}_{K_i}$ then $\sigma_j(K_i)$ is self-normalising in T_j.

Suppose now that T is as in row 4 of Table 3 and set

$$\underline{K}_i = \prod_{X \in \mathcal{X}_{K_i}} X \times \prod_{T_j \notin \bigcup_{X \in \mathcal{X}_{K_i}} \operatorname{Supp} X} \sigma_j(K_i)'.$$

Then it follows from Proposition 7.2 that $\underline{K}_i \leqslant K_i$, and from the definition of $\overline{K}_i$ that $K_i \leqslant \overline{K}_i$. Now each strip $X \in \mathcal{X}_{K_i}$ is self-normalising in $\sigma_{\operatorname{Supp} X}(M)$, and $\mathbb{N}_{T_j}\ (\sigma_j(K_i)') = \mathbb{N}_{T_j}\ (\sigma_j(K_i)) = \sigma_j(K_i)$ whenever $T_j \notin \operatorname{Supp} X$ for some $X \in \mathcal{X}_{K_i}$. Hence $\mathbb{N}_M\ (\underline{K}_i) = \overline{K}_i$ and $\mathbb{N}_M\ (K_i) \leqslant \overline{K}_i$. On the other hand, Lemma 3.4 implies that $\mathbb{N}_M\ (\underline{K}_i) \leqslant \mathbb{N}_M\ (K_i)$, and so $\overline{K}_i = \mathbb{N}_M\ (\underline{K}_i) = \mathbb{N}_M\ (K_i)$.

It remains to prove that $K_i' = \underline{K}_i$. As for $j = 1, \ldots, k$, either $\sigma_j(K_i)' = \sigma_j(K_i)$ or $\sigma_j(K_i)/\sigma_j(K_i)'$ is isomorphic to $\mathbb{Z}_2$, it follows that $K_i/\underline{K}_i$ is an elementary abelian 2-group. Thus $K_i' \leqslant \underline{K}_i$. On the other hand, $\underline{K}_i$ is a direct product of non-abelian, finite simple groups, and so no quotient of $\underline{K}_i$ is abelian. This proves that $K_i' = \underline{K}_i$ as required. ∎

Suppose that G_1, G_2 are groups and let H be a subgroup of G_1. If $\alpha\colon H \to G_2$ is an injective homomorphism then we define

$$\mathsf{Diag}\,\alpha = \{(h, \alpha(h)) |\ h \in H\}$$

as a subgroup of $G_1 \times G_2$.

Lemma 7.4: *Let $\mathcal{E} \in \mathsf{CD}_{1\mathrm{S}}(G)$ and let $\mathcal{K}_\omega(\mathcal{E}) = \{K_1, \ldots, K_\ell\}$. Then*

$$\mathbb{N}_M\left(M_\omega\right) = \mathbb{N}_M\left(K_1\right) \cap \cdots \cap \mathbb{N}_M\left(K_\ell\right),$$

and $\mathbb{N}_M\left(M_\omega\right)$ is the direct product of $k/2$ strips of length 2 in M. Moreover, if Y is a strip in $\mathbb{N}_M\left(M_\omega\right)$ then Y is isomorphic to the group in the last column of the appropriate row of Table 3.

Proof: For $i = 1, \ldots, \ell$ we have $K_i' \leqslant K_i \leqslant \mathbb{N}_M\left(K_i\right)$ with equality if T is as in one of the rows 1–3 of Table 3 (see Lemma 7.3). Let $X_1, \ldots, X_{k/2}$ be the strips involved in $\mathcal{K}_\omega(\mathcal{E})$. Then Theorem 2.2 implies that $\{\mathrm{Supp}\, X_1, \ldots, \mathrm{Supp}\, X_{k/2}\}$ is a partition of $\{T_1, \ldots, T_k\}$, and it follows from Lemma 7.3 that, for $i = 1, \ldots, \ell$,

$$K_i' = \prod_{j=1}^{k/2} \sigma_{\mathrm{Supp}\, X_j}(K_i') \quad \text{and} \quad \mathbb{N}_M\left(K_i\right) = \prod_{j=1}^{k/2} \sigma_{\mathrm{Supp}\, X_j}(\mathbb{N}_M\left(K_i\right)).$$

Let $\underline{M}_\omega = K_1' \cap \cdots \cap K_\ell'$. Set $M_0 = \overline{K}_1 \cap \cdots \cap \overline{K}_\ell$ and recall that $M_0 = \mathbb{N}_M\left(K_1\right) \cap \cdots \cap \mathbb{N}_M\left(K_\ell\right)$. Then

$$\underline{M}_\omega = \prod_{j=1}^{k/2} \sigma_{\mathrm{Supp}\, X_j}(\underline{M}_\omega) \quad \text{and} \quad M_0 = \prod_{j=1}^{k/2} \sigma_{\mathrm{Supp}\, X_j}(M_0).$$

For $i = 1, \ldots, k/2$, the subgroup $X_i = \mathsf{Diag}\,\alpha_i$ for some isomorphism $\alpha_i\colon T_{\min X_i} \to T_{\max X_i}$. Let K_{j_1} and K_{j_2} be the elements of the Cartesian system such that $\sigma_{\min X_i}(K_{j_1}) \neq T_{\min X_i}$ and $\sigma_{\max X_i}(K_{j_2}) \neq T_{\max X_i}$. Set

$$\widehat{Y}_i = \sigma_{\min X_i}(K_{j_1}) \cap \alpha_i^{-1}(\sigma_{\max X_i}(K_{j_2}))$$

and

$$\check{Y}_i = \sigma_{\min X_i}(K_{j_1})' \cap \alpha_i^{-1}(\sigma_{\max X_i}(K_{j_2}))'.$$

Let $\widehat{\alpha}_i$ and $\check{\alpha}_i$ denote the restrictions of α_i to the subgroups $\widehat{Y}_i$ and $\check{Y}_i$, respectively. Then we have that

$$\sigma_{\mathrm{Supp}\,X_i}(M_0) = \mathsf{Diag}\,\widehat{\alpha}_i \quad \text{and} \quad \sigma_{\mathrm{Supp}\,X_i}(\underline{M}_\omega) = \mathsf{Diag}\,\check{\alpha}_i.$$

Suppose first that T is as in rows 1–3 in Table 3. Then, as $M_\omega = \underline{M}_\omega = M_0$, it follows that M_ω is the direct product of the $\mathsf{Diag}\,\widehat{\alpha}_i$. On the other hand, by [Lemma 2.1, PS02], the factorisation

$$T_{\min X_i} = \sigma_{\min X_i}(K_{j_1})\alpha_i^{-1}(\sigma_{\max X_i}(K_{j_2}))$$

involves isomorphic subgroups. Thus the subgroups involved in this factorisation must be as in [Lemma 5.2, BPS04]. Now the isomorphism type of the intersection $\widehat{Y}_i$ can be determined using the [Atlas] in rows 1–2 and [3.1.1(vi), Kle87] in row 3. Hence we find that, for T in one of these rows, the group $\widehat{Y}_i$ is isomorphic to the subgroup in the last column of Table 3. By [Lemma 5.2, BPS04], $\widehat{Y}_i$ is self-normalising with trivial centraliser in $T_{\min X_i}$, and so Lemma 3.5 implies that $\mathbb{N}_{T_{\min X_i} \times T_{\max X_i}}(\mathsf{Diag}\,\widehat{\alpha}) = \mathsf{Diag}\,\widehat{\alpha}$, and so $\mathbb{N}_M(M_\omega) = M_\omega$, as required.

Suppose now that T is as in row 4 of Table 3. Then the isomorphisms $\widehat{Y}_i \cong \mathsf{D}_{q^2+1}\cdot 2$ and $\check{Y}_i \cong \mathsf{D}_{q^2+1}$ follow from [3.2.1(d), LPS90]. Using Lemma 3.6 we obtain that $\mathbb{N}_{T_{\min X_i}}(\check{Y}_i) = \mathbb{N}_{T_{\min X_i}}(\widehat{Y}_i) = \widehat{Y}_i$ and $\mathbb{C}_{T_{\min X_i}}(\check{Y}_i) = \mathbb{C}_{T_{\min X_i}}(\widehat{Y}_i) = 1$. Thus Lemma 3.5 implies that $\mathbb{N}_M(\underline{M}_\omega) = M_0$, $\underline{M}_\omega \cong (\mathsf{D}_{q^2+1})^{k/2}$, $M_0 \cong (\mathsf{D}_{q^2+1}\cdot 2)^{k/2}$, and $M_0/\underline{M}_\omega$ is an elementary abelian 2-group. Hence $\mathbb{N}_M(M_\omega) \leqslant M_0$. On the other hand Lemma 3.4 implies that $M_0 \leqslant \mathbb{N}_M(M_\omega)$. Therefore $\mathbb{N}_M(M_\omega) = M_0$, as required. ∎

Now we can prove Theorem 7.1.

7.2. Proof of Theorem 7.1. **Prop1S[a]** For each i there is a unique j such that $\sigma_i(K_j) < T_i$. Thus the $\overline{K}_j$ are proper subgroups of M, and no T_i is contained in $\overline{M}_\omega$. Hence M acts faithfully on $\overline{\Omega}$. Lemmas 7.3 and 7.4 imply that $\overline{K}_1 \cap \cdots \cap \overline{K}_\ell = \mathbb{N}_M(M_\omega) = \overline{M}_\omega$. Therefore (1) holds for $\overline{\mathcal{K}}_\omega(\mathcal{E})$. As (2) holds for $\mathcal{K}_\omega(\mathcal{E})$, and for $i = 1, \ldots, \ell$, $K_i \leqslant \overline{K}_i$, we have that (2) also holds for $\overline{\mathcal{K}}_\omega(\mathcal{E})$. Therefore $\overline{\mathcal{K}}_\omega(\mathcal{E})$ is a Cartesian system of subgroups for M with respect to $\overline{\omega}$. We claim that $\overline{\mathcal{K}}_\omega(\mathcal{E})$ is invariant under conjugation by $\overline{G_{\overline{\omega}}}$. Note that $G_{\overline{\omega}} = M_{\overline{\omega}}G_\omega$, and so it suffices to prove that $\overline{\mathcal{K}}_\omega(\mathcal{E})$ is invariant under conjugation by G_ω. This, however, follows from the fact that $\{K_1, \ldots, K_\ell\}$ is G_ω-invariant and, by Lemma 7.3, $\overline{\mathcal{K}}_\omega(\mathcal{E}) = \{\mathbb{N}_M(K_1), \ldots, \mathbb{N}_M(K_\ell)\}$.

Prop1S[b] It follows from Theorem 2.2 that Prop1S[b](i) holds. It is clear that Prop1S[b](ii) also holds. Recall that $\overline{M}_\omega = M_{\overline{\omega}} = \mathbb{N}_M(M_\omega)$. Let X be a non-trivial, full strip involved in K_i, and let $j_1,\ j_2 \in \{1,\dots,\ell\}\setminus\{i\}$ be such that $\sigma_{\min X}(K_{j_1}) < T_{\min X}$ and $\sigma_{\max X}(K_{j_2}) < T_{\max X}$. Set $A = \sigma_{\min X}(K_{j_1})$ and $B = \sigma_{\max X}(K_{j_2})$. Suppose that $g \in G_\omega$ is such that $T_{\min X}^g = T_{\max X}$. Then $A^g = \sigma_{\min X}(K_{j_1})^g = \sigma_{\max X}(K_{j_1}^g)$. As j_2 is the unique integer such that $\sigma_{\max X}(K_{j_2}) < T_{\max X}$, we obtain that $K_{j_1}^g = K_{j_2}$, and so $A^g = B$. Hence Prop1S[b](iii) also holds. Note that, as $\mathcal{K}_\omega(\mathcal{E})$ is a Cartesian system, we have that $K_i(K_{j_1}\cap K_{j_2}) = M$. Thus

$$T_{\min X}\times T_{\max X} = \sigma_{\mathrm{Supp}\,X}(K_j)\sigma_{\mathrm{Supp}\,X}(K_{j_1}\cap K_{j_2}).$$

As $\sigma_{\mathrm{Supp}\,X}(K_{j_1}\cap K_{j_2}) \leqslant A\times B$ we obtain that $T_{\min X}\times T_{\max X} = X(A\times B)$. Since each $\overline{K}_j$ is the direct product, over $X_i\in\mathcal{X}$, of its projection under $\sigma_{\mathrm{Supp}\,X_i}$, so is the subgroup $\overline{M}_\omega$. Thus $X\cap(A\times B) = \sigma_{\mathrm{Supp}\,X}(\overline{M}_\omega) = \sigma_{\mathrm{Supp}\,X}(\mathbb{N}_M\,(M_\omega))$, by Lemma 7.4. Therefore Prop1S[b](iv) holds. Let us now prove Prop1S[b](v). As $A\times B\leqslant T_{\min X}\times T_{\max X}$, $X\leqslant T_{\min X}\times T_{\max X}$, and $\{T_{\min X}, T_{\max X}\}$ is a block for the G_ω-action on $\mathcal{T}$, it follows that

$$\mathbb{N}_{G_\omega}\,(A\times B)\leqslant \mathbb{N}_{G_\omega}\,(T_{\min X}\times T_{\max X})$$

and

$$\mathbb{N}_{G_\omega}\,(X)\leqslant \mathbb{N}_{G_\omega}\,(T_{\min X}\times T_{\max X})\,.$$

Let $g\in\mathbb{N}_{G_\omega}\,(T_{\min X}\times T_{\max X})$. Then X^g is a strip involved in $K_i^g\in\mathcal{K}_\omega(\mathcal{E})$ such that X and X^g have the same support. Hence Theorem 2.2 implies that $X = X^g$, and so $g\in\mathbb{N}_{G_\omega}\,(X)$. Also, the element g either normalises both subgroups $T_{\min X}$ and $T_{\max X}$ or swaps these two subgroups. Hence one of the following scenario holds: either

$$\sigma_{\min X}(K_{j_1})^g = \sigma_{\min X}(K_{j_1}^g)\quad\text{and}\quad\sigma_{\max X}(K_{j_2})^g = \sigma_{\max X}(K_{j_2}^g);$$

or

$$\sigma_{\min X}(K_{j_1})^g = \sigma_{\max X}(K_{j_1}^g)\quad\text{and}\quad\sigma_{\max X}(K_{j_2})^g = \sigma_{\min X}(K_{j_2}^g).$$

Since K_{j_1} and K_{j_2} are the unique elements of $\mathcal{K}_\omega(\mathcal{E})$ whose projection to $T_{\min X}$ and $T_{\max X}$, respectively, are proper, we obtain that $\{A^g, B^g\} = \{A, B\}$. Therefore $(A\times B)^g = A\times B$, and so $g\in\mathbb{N}_{G_\omega}\,(A\times B)$.

Prop1S[c] First we prove that G_ω induces a group of automorphisms of the graph $\Gamma(G,\mathcal{E})$. Suppose that, for some $K\in\mathcal{K}_\omega(\mathcal{E})$ and $X\in\mathcal{X}$, the edge $\{K, X\}$ is in E_1 and $g\in G_\omega$. Then $\sigma_{\min X}(K) < T_{\min X}$ or $\sigma_{\max X}(K) < T_{\max X}$. Suppose without loss of generality that $\sigma_{\min X}(K) < T_{\min X}$. Then $\sigma_m(K^g) < T_m$

where $m \in \{1,\dots,k\}$ is such that $T^g_{\min X} = T_m$. As X^g covers T_m, it follows that $\{K,X\}^g = \{K^g,X^g\} \in E_1$. Now let $\{K,X\} \in E_2$. Then X is involved in K and hence X^g is involved in K^g, whence $\{K^g,X^g\} \in E_2$. Thus G_ω preserves adjacency in $\Gamma(G,\mathcal{E})$. Moreover, under the conjugation action of G_ω, the sets $\mathcal{K}_\omega(\mathcal{E})$ and $\mathcal{X}$ are G_ω-orbits. We claim that E_1 and E_2 are also G_ω-orbits. Suppose that $\{K_1,X_1\},\{K_2,X_2\} \in E_1$. There exist i_1, i_2 such that $T_{i_1} \in \operatorname{Supp} X_1$, $T_{i_2} \in \operatorname{Supp} X_2$, $\sigma_{i_1}(K_1) < T_{i_1}$ and $\sigma_{i_2}(K_2) < T_{i_2}$. Since G_ω is transitive on $T_1,\dots,T_k$, there is an element $g \in G_\omega$ such that $T^g_{i_1} = T_{i_2}$. Then $T_{i_2} \in \operatorname{Supp} X^g_1 \cap \operatorname{Supp} X_2$, and so Theorem 2.2 implies that $X^g_1 = X_2$. We also have that $\sigma_{i_1}(K_1)^g = \sigma_{i_2}(K^g_1) < T_{i_2}$. Since K_2 is the unique element in $\mathcal{K}_\omega(\mathcal{E})$ with proper projection in T_{i_2} we have $K^g_1 = K_2$. Thus $\{K_1,X_1\}^g = \{K_2,X_2\}$, and so G_ω is transitive on E_1. Now let $\{K_1,X_1\},\{K_2,X_2\} \in E_2$. Then X_1 is involved in K_1 and X_2 is involved in K_2. There is an element $g \in G_\omega$ such that $X^g_1 = X_2$, which implies that $K^g_1 = K_2$. Thus $\{K_1,X_1\}^g = \{K_2,X_2\}$, and G_ω is transitive on E_2. Finally, suppose that the elements of $\mathcal{X}$ have E_1-valency 2 and let $\{K,X\} \in E_1$. Suppose without loss of generality that $\sigma_{\min X}(K) < T_{\min X}$ and let $g \in (G_\omega)_{\{K,X\}}$. Then $\{T_{\min X}, T_{\max X}\}^g = \{T_{\min X}, T_{\max X}\}$, and so either

$$T^g_{\min X} = T_{\min X} \quad \text{and} \quad T^g_{\max X} = T_{\max X}$$

or

$$T^g_{\min X} = T_{\max X} \quad \text{and} \quad T^g_{\max X} = T_{\min X}$$

In the latter case we would have $\sigma_{\min X}(K)^g = \sigma_{\max X}(K)$. As X has E_1-valency 2, we have that $\sigma_{\max X}(L) < T_{\max X}$ for a unique $L \in \mathcal{K}_\omega(\mathcal{E})$ and this L is different from K, which is a contradiction. Thus $T^g_{\min X} = T_{\min X}$ and $T^g_{\max X} = T_{\max X}$ must hold. Therefore $g \in \mathbb{N}_{G_\omega}(T_{\min X}) \cap \mathbb{N}_{G_\omega}(T_{\max X})$. Conversely suppose that $g \in \mathbb{N}_{G_\omega}(T_{\min X}) \cap \mathbb{N}_{G_\omega}(T_{\max X})$. Then clearly $g \in \mathbb{N}_{G_\omega}(X)$. Moreover, $\sigma_{\min X}(K)^g = \sigma_{\min X}(K^g)$, and since K is the unique element of $\mathcal{K}_\omega(\mathcal{E})$ such that $\sigma_{\min X}(K) < T_{\min X}$, it follows that $K^g = K$. Therefore $\{K,X\}^g = \{K,X\}$. Hence property Prop1S[c] holds.

Prop1S[d] By Proposition 7.2 and Lemma 7.4 the groups T, A_i (which is G_ω-conjugate to B_i), and $\sigma_{\operatorname{Supp} X_i}(M_{\overline{\omega}})$ are as in Table 3. By Lemma 7.4, the group $\overline{M}_\omega$ is a direct product of pairwise disjoint strips, and each such strip Y is self-normalising in $\sigma_{\operatorname{Supp} Y}(M)$ (see Lemma 3.5). Thus $\overline{M}_\omega$ is a self-normalising subgroup of M. Hence [Theorem 4.2A, DM96] implies that $\mathbb{C}_{\mathsf{Sym}\,\overline{\Omega}}(M) = 1$. Thus $\overline{G}$ can be embedded into $\mathsf{Aut}\, M$, and so $\overline{G}$ is quasiprimitive and M is its unique minimal normal subgroup.

By Lemma 7.3, if one of the rows 1–3 of Table 3 is valid, then $K_i = \overline{K}_i$ for all i, and so $M_\omega = \overline{M}_\omega$. Thus the sets $\overline{\Omega}$ and Ω can be identified naturally, and the groups G and $\overline{G}$ are naturally permutationally isomorphic.

If T is as in row 4 of Table 3, then it follows from Lemma 7.4 that $\mathbb{N}_M(M_\omega)/M_\omega$ is an elementary abelian 2-group with rank at most $k/2$. Thus each block in $\overline{\Omega}$ has size dividing $k/2$. Further, it follows from [Theorem 4.2A, DM96] that $\mathbb{C}_{\mathsf{Sym}\,\Omega}(M)$ is also an elementary abelian 2-group of rank at most $k/2$. As M is a minimal normal subgroup of G and is faithful on Ω, we obtain that $N \cap M = 1$, and so $N \leqslant \mathbb{C}_{\mathsf{Sym}\,\Omega}(M)$. Thus N an elementary abelian 2-group of rank at most $k/2$. ∎

7.3. A converse of Theorem 7.1. Theorem 7.1 can be reversed in the following sense. Suppose that G is an innately transitive group on Ω with non-abelian plinth M, and let $T_1, \ldots, T_k$ be the simple normal subgroups of M. Let $\omega \in \Omega$. Assume that M_ω is a direct product of pairwise disjoint strips, each of length 2. Let $\mathcal{Y}$ denote the set of strips involved in M_ω, say $\mathcal{Y} = \{Y_1, \ldots, Y_{k/2}\}$. Let X_1, A_1, B_1 be subgroups of M, and let $\Gamma = (V \cup \mathcal{Y}, E_1 \cup E_2)$ be a bipartite graph satisfying properties Prop1S[b] and Prop1S[c], that is, the following all hold:

(i) X_1 is a full strip of length 2;
(ii) A_1 is a proper subgroup of $T_{\min X_1}$ and B_1 is a proper subgroup of $T_{\max X_1}$;
(iii) A_1 and B_1 are conjugate under G_ω;
(iv) $X_1(A_1 \times B_1) = T_{\min X_1} \times T_{\max X_1}$, $X_1 \cap (A_1 \times B_1) = \sigma_{\mathrm{Supp}\,X_1}(M_\omega)$;
(v) $\mathbb{N}_{G_\omega}(T_{\min X_1} \times T_{\max X_1}) = \mathbb{N}_{G_\omega}(X_1) = \mathbb{N}_{G_\omega}(A \times B)$;

and also the group G_ω induces a group of automorphisms of the bipartite graph Γ, such that V, $\mathcal{Y}$, E_1 and E_2 are G_ω-orbits. Further, each element of $\mathcal{Y}$ is adjacent to one edge or two edges from E_1, and one edge of E_2. If the elements of $\mathcal{Y}$ are adjacent to two elements of E_1 then the following must also hold: if $\{v, Y\} \in E_1$, for some $v \in V$ and $Y \in \mathcal{Y}$, then $(G_\omega)_{\{v,Y\}} = \mathbb{N}_{G_\omega}(T_{\min Y}) \cap \mathbb{N}_{G_\omega}(T_{\max Y})$.

We claim that $\mathrm{Supp}\,X_1 = \mathrm{Supp}\,Y_j$ for some $j \in \{1, \ldots, k/2\}$. If this is not true then we have $\sigma_{\mathrm{Supp}\,X_1}(M_\omega) = \sigma_{T_{\min X_1}}(M_\omega) \times \sigma_{T_{\max X_1}}(M_\omega)$, and by property (iv) above, this is contained in $X_1 \cap (A_1 \times B_1)$, which is a contradiction. Hence we may assume, without loss of generality, that $\mathrm{Supp}\,X_1 = \mathrm{Supp}\,Y_1$. Let $v_1, v_2 \in V$ be such that $\{v_1, Y_1\} \in E_1$ and $\{v_2, Y_1\} \in E_2$, and define

$$K_{v_1,Y_1} = \begin{cases} A_1 \times T_{\max Y_1} & \text{if } Y_1 \text{ has } E_1\text{-valency } 2; \\ A_1 \times B_1 & \text{otherwise.} \end{cases}$$

For $v \in V$ and $Y \in \mathcal{Y}$, define

$$K_{v,Y} = \begin{cases} (K_{v_1,Y_1})^g & \text{if } \{v,Y\} \in E_1 \text{ and } g \in G_\omega \text{ is such that } \{v_1,Y_1\}^g = \{v,Y\}; \\ X_1^g & \text{if } \{v,Y\} \in E_2 \text{ and } g \in G_\omega \text{ is such that } \{v_2,Y_1\}^g = \{v,Y\}; \\ T_{\min Y} \times T_{\max Y} & \text{otherwise.} \end{cases}$$

We claim that the definition of $K_{v,Y}$ is independent of the chosen $g \in G_\omega$. Indeed, if two elements $g_1,\ g_2 \in G_\omega$ are such that $\{v_1,Y_1\}^{g_1} = \{v_1,Y_1\}^{g_2} = \{v,Y\}$, then $g_1g_2^{-1} \in (G_\omega)_{\{v_1,Y_1\}}$. If Y_1 has E_1-valency 2 then by assumption,

$$g_1g_2^{-1} \in \mathbb{N}_{G_\omega}(T_{\min Y_1}) \cap \mathbb{N}_{G_\omega}(T_{\max Y_1}) = \mathbb{N}_{G_\omega}(A_1) \cap \mathbb{N}_{G_\omega}(B_1).$$

Hence

$$\begin{aligned}(K_{v_1,Y_1})^{g_1g_2^{-1}} &= (A_1 \times T_{\max Y_1})^{g_1g_2^{-1}} = A_1^{g_1g_2^{-1}} \times (T_{\max Y_1})^{g_1g_2^{-1}} = A_1 \times T_{\max Y_1} \\ &= K_{v_1,Y_1}\end{aligned}$$

If Y_1 has E_1-valency 1 then, as $g_1g_2^{-1} \in \mathbb{N}_{G_\omega}(T_{\min Y_1} \times T_{\max Y_1})$, property (v) implies that

$$(K_{v_1,Y_1})^{g_1g_2^{-1}} = (A_1 \times B_1)^{g_1g_2^{-1}} = A_1 \times B_1 = K_{v_1,Y_1}.$$

Thus $(K_{v_1,Y_1})^{g_1} = (K_{v_1,Y_1})^{g_2}$, as claimed. Similarly if $\{v_2,Y_1\}^{g_1} = \{v_2,Y_1\}^{g_2}$ for some $g_1,\ g_2 \in G_\omega$ then

$$g_1g_2^{-1} \in \mathbb{N}_{G_\omega}(T_{\min Y_1} \times T_{\max Y_1}) = \mathbb{N}_{G_\omega}(X_1)$$

and so $X_1^{g_1} = X_1^{g_2}$. So also in this case the definition of $K_{v,Y}$ is independent of the chosen element g.

We now claim that for each $v \in V$, $Y \in \mathcal{Y}$, and $g \in G_\omega$ we have

$$K_{v^g,Y^g} = (K_{v,Y})^g. \tag{14}$$

Indeed suppose that $\{v,Y\} \in E_1$. Then there is an element $g_1 \in G_\omega$ such that $\{v_1,Y_1\}^{g_1} = \{v,Y\}$ and so $\{v^g,Y^g\} = \{v_1,Y_1\}^{g_1g}$. Now K_{v^g,Y^g} was defined above as $(K_{v_1,Y_1})^{g_1g}$, and $K_{v,Y}$ was defined as $(K_{v_1,Y_1})^{g_1}$. Thus (14) holds in this case. Similarly, if $\{v,Y\} \in E_2$ then there is an element $g_1 \in G_\omega$ such that $\{v_2,Y_1\}^g = \{v,Y\}$, and the same argument shows that (14) holds. Finally, if $\{v,Y\} \notin E_1 \cup E_2$ then $\{v^g,Y^g\} \notin E_1 \cup E_2$. Hence $K_{v,Y} = T_{\min Y} \times T_{\max Y}$ and $K_{v^g,Y^g} = T_{\min Y^g} \times T_{\max Y^g}$. As $\operatorname{Supp} Y^g = (\operatorname{Supp} Y)^g$ we have that $(T_{\min Y} \times T_{\max Y})^g = T_{\min Y^g} \times T_{\max Y^g}$. Hence (14) holds in all cases.

Now we note that $K_{v,Y}$ is a subgroup of $T_{\min Y} \times T_{\max Y}$, and, as the elements of $\mathcal{Y}$ are pairwise disjoint strips, we can define

$$K_v = \prod_{Y \in \mathcal{Y}} K_{v,Y}$$

and set $\mathcal{K} = \{K_v |\ v \in V\}$.

Our next task is to prove that $\mathcal{K}$ is a Cartesian system, that is, equations (1) and (2) hold. To help ourselves with this, first we prove analogous properties for the subgroups $K_{v,Y}$ of the K_v. If Y_1 has E_1-valency 1 then, Y_1 is adjacent to two vertices v_1 and v_2 and, by property (iv), we have

$$K_{v_1,Y_1} K_{v_2,Y_1} = (A_1 \times B_1)X_1 = T_{\min Y_1} \times T_{\max Y_1} \tag{15}$$

and

$$K_{v_1,Y_1} \cap K_{v_2,Y_1} = (A_1 \times B_1) \cap X_1 = \sigma_{\mathrm{Supp}\, Y_1}(M_\omega). \tag{16}$$

Suppose now that Y_1 has E_1-valency 2, and let v_3 be a vertex such that $v_1 \neq v_3$ and $\{v_3, Y_1\} \in E_1$. Then there is some element $g \in G_\omega$ such that $A^g = B$. Then we must have that $T^g_{\min Y_1} = T_{\max Y_1}$, and so, by the conditions above, g must interchange v_1 and v_3. Then the argument above shows that $K_{v_3,Y_1} = T_{\min Y_1} \times B$. Thus, by (iv),

$$\{K_{v_1,Y_1}, K_{v_2,Y_1}, K_{v_3,Y_1}\} \text{ is a strong multiple factorisation of } T_{\min Y_1} \times T_{\max Y_1} \tag{17}$$

and

$$K_{v_1,Y_1} \cap K_{v_2,Y_1} \cap K_{v_3,Y_1} = \sigma_{\mathrm{Supp}\, Y_1}(M_\omega). \tag{18}$$

Now we are ready to show that (1) and (2) hold. As M_ω and the elements of $\mathcal{K}$ are direct products of their projections under $\sigma_{\mathrm{Supp}\, Y}$, for $Y \in \mathcal{Y}$, it suffices to prove that

$$\bigcap_{v \in V} K_{v,Y} = \sigma_{\mathrm{Supp}\, Y}(M_\omega) \quad \text{and} \quad K_{v,Y}\left(\bigcap_{v' \neq v} K_{v',Y}\right) = T_{\min Y} \times T_{\max Y}$$

holds for all $Y \in \mathcal{Y}$. If $g \in G_\omega$ such that $Y_1^g = Y$ then, by (16) and (18), we have that

$$\bigcap_{v \in V} K_{v,Y} = X_1^g \cap (A_1 \times B_1)^g = (X_1 \cap (A_1 \times B_1))^g = \sigma_{\mathrm{Supp}\, Y_1}(M_\omega)^g$$
$$= \sigma_{\mathrm{Supp}\, Y}(M_\omega).$$

Also, using (15) and (17),

$$K_{v,Y}\Big(\bigcap_{v'\neq v} K_{v',Y}\Big) = \Big(K_{v^{g^{-1}},Y_1}\Big(\bigcap_{v'\neq v} K_{v'^{g^{-1}},Y_1}\Big)\Big)^g$$
$$= (T_{\min Y_1} \times T_{\max Y_1})^g = T_{\min Y} \times T_{\max Y}.$$

Hence (1) and (2) hold.

It remains to prove that $\mathcal{K}$ is G_ω-invariant. Let $v \in V$ and $g \in G_\omega$. Then

$$K_v^g = \Big(\prod_{Y\in\mathcal{Y}} K_{v,Y}\Big)^g = \prod_{Y\in\mathcal{Y}} K_{v^g,Y^g} = K_{v^g}.$$

Thus $K_v^g \in \mathcal{K}$. Therefore $\mathcal{K}$ is a G_ω-invariant Cartesian system of subgroups in M. It also follows from the last displayed equation that the actions of G_ω on $\mathcal{K}$ and on V are equivalent, and so G_ω is transitive on $\mathcal{K}$. It follows from the definition of the K_v that $\mathcal{E}(\mathcal{K}) \in \mathsf{CD}_{1\mathrm{S}}(G)$ and that $\Gamma = \Gamma(G, \mathcal{E})$.

Suppose that G is an innately transitive group with a non-abelian plinth M and a point stabiliser M_ω is a direct product of pairwise disjoint strips with length 2 such that the isomorphism types of these groups are as prescribed by Table 3. Then, as in the previous sections, it is sometimes possible to describe the elements of $\mathsf{CD}_{1\mathrm{S}}(G)$ via studying the action of G_ω on the set of strips in M_ω. This phenomenon is illustrated in the following example.

Example 7.5: Suppose that $\Gamma = (V_1 \cup V_2, E_1 \cup E_2)$ is a bipartite graph with an automorphism group A satisfying the Combinatorial Property where V_1, V_2 and A play the rôle of $\mathcal{K}_\omega(\mathcal{E})$, $\mathcal{X}$, and G_ω, respectively. Suppose that V_2 has 4 vertices and A induces a group isomorphic to A_4 on V_2. As each vertex in V_2 is adjacent to exactly one edge in E_2 it follows that V_1 must have 2 or 4 vertices. If V_1 has 2 vertices, v_1 and v_2 say, then the set of vertices in V_2 connected to v_1 via E_2 is a block for the action of A. Such a block would have 2 elements, and this is impossible, as A_4 has no non-trivial blocks. Hence $|V_1| = 4$. If a vertex of V_2 is adjacent with 2 edges in E_1 then E_1 must have 8 elements. Thus E_1 cannot be an A-orbit, as $|A|$ is not divisible by 8. Thus each element of V_2 must be adjacent with exactly one edge of E_1. Suppose without loss of generality that $\{v_1, u_1\} \in E_1$ and $\{v_2, u_1\} \in E_2$ for some $u_1 \in V_2$. Then $v_1 \neq v_2$, and so $A_{u_1} \leqslant A_{v_1} \cap A_{v_2} = 1$. This is a contradiction since $|A : A_{u_1}| = |V_1| = 4$. Thus E_2 cannot have 4 elements, and so no such graph Γ exists.

This simple graph theoretic argument shows that if G is an innately transitive group with plinth $M = T_1 \times \cdots \times T_8$ such that a point stabiliser M_ω is the direct product of 4 pairwise disjoint strips and G_ω induces a group permutationally isomorphic to A_4 on these strips then $\mathsf{CD}_{1S}(G) = \emptyset$.

References

[BP98] R. W. Baddeley and C. E. Praeger, *On classifying all full factorisations and multiple-factorisations of the finite almost simple groups,* Journal of Algebra **204** (1998), 129–187.

[BP03] R. W. Baddeley and C. E. Praeger, *On primitive overgroups of quasiprimitive permutation groups,* Journal of Algebra **263** (2003), 294–344.

[BPS04] R. W. Baddeley, C. E. Praeger and C. Schneider, *Transitive simple subgroups of wreath products in product action,* Journal of the Australian Mathematical Society **77** (2004), 55–72.

[BPS06] R. W. Baddeley, C. E. Praeger and C. Schneider, *Innately transitive subgroups of wreath products in product action,* Transactions of the American Mathematical Society **358** (2006), 1619–1641.

[BPSxx] R. W. Baddeley, C. E. Praeger and C. Schneider, *Intransitive Cartesian decompositions preserved by innately transitive groups,* Transactions of the American Mathematical Society, to appear. arxiv.org/math.GR/0405241.

[BPSyy] R. W. Baddeley, C. E. Praeger and C. Schneider, *Quasiprimitive groups and blow-up decompositions,* Journal of Algebra, to appear. arXiv.org/math.GR/0409449.

[BamP04] J. Bamberg and C. E. Praeger, *Finite permutation groups with a transitive minimal normal subgroup,* Proceedings of the London Mathematical Society (3) **89** (2004), 71–103.

[Atlas] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups,* Oxford University Press, Oxford, 1985.

[DM96] J. D. Dixon and B. Mortimer, *Permutation Groups,* Springer-Verlag, New York, 1996.

[Kle87] P. B. Kleidman, *The maximal subgroups of the finite* 8*-dimensional orthogonal groups* $P\Omega_8^+(q)$ *and of their automorphism groups,* Journal of Algebra **110** (1987), 173–242.

[Kov89a] L. G. Kovács, *Primitive subgroups of wreath products in product action,* Proceedings of the London Mathematical Society (3) **58** (1989), 306–322.

[Kov89b] L. G. Kovács, *Wreath decompositions of finite permutation groups,* Bulletin of the Australian Mathematical Society **40** (1989), 255–279.

[LPS88] M. W. Liebeck, C. E. Praeger and J. Saxl, *On the O'Nan-Scott Theorem for finite primitive permutation groups,* Journal of the Australian Mathematical Society Ser. A **44** (1988), 389–396.

[LPS90] M. W. Liebeck, C. E. Praeger and J. Saxl, *The maximal factorizations of the finite simple groups and their automorphism groups,* Memoirs of the American Mathematical Society **86(432)** (1990), iv+151.

[Pra90] C. E. Praeger, *The inclusion problem for finite primitive permutation groups,* Proceedings of the London Mathematical Society (3) **60** (1990), 68–88.

[Pra93] C. E. Praeger, *An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to* 2*-arc transitive graphs,* Journal of the London Mathematical Society (2) **47** (1993), 227–239.

[PS02] C. E. Praeger and C. Schneider, *Factorisations of characteristically simple groups,* Journal of Algebra **255** (2002), 198–220.

[PS03] C. E. Praeger and C. Schneider, *Ordered triple designs and wreath products of groups,* in *Statistics and Science: A Festschrift for Terry Speed* (Darlene R. Goldstein, ed.), Institute of Mathematical Statistics, Lecture Notes — Monograph Series, volume **40**, 2003, pp. 103–113.